Synchronisation of clocks

Not only are clocks slowed by motion through the stationary system, but their synchronisation is affected. Clocks can be accurately synchronised in two ways. One is to physically move one clock next to the other and do it electronically by sending a signal from one to the other down a short wire, then return the synchronised clock to its required location. The other is to send a radio signal from one clock to the other, and allow for the time taken for the radio signal to reach it. Both are affected by motion through the stationary system. As a clock is moved around within the moving system, its velocity through the stationary system varies affecting the rate of the clock and causing synchronisation errors. Alternatively, using radio signals, the actual distance travelled through the stationary system by the radio signal is different from the distance measured within the moving system.

The amazing thing is that both factors give exactly the same result.

Moving a clock

A clock is taken on a journey within the moving system. While is is moving within the moving system, its velocity through the stationary system is changed altering the extent to which it runs slow. The difference in clock rates is:

$$\frac{dt'}{dt} = \sqrt{1 - \frac{v^2}{c^2}} - \sqrt{1 - \frac{(v + w_x)^2 + w_y^2 + w_z^2}{c^2}}$$

There is no exact analysis of this, but we can on the assumption that $w \ll v \ll c$ expand each of the square roots into a series with regard to v and perform the subtraction. Omitting higher powers gives:

$$\frac{dt'}{dt} = \frac{2v\,w_x + w^2}{2c^2} + \ldots = \frac{v\,w_x}{c^2} \qquad \text{for} \qquad w \ll v$$

The loss of time is:

$$\delta t' = \int \frac{v\,w_x}{c^2}\,dt = \frac{v}{c^2}\int w_x\,dt = \frac{v\,x_m}{c^2}$$

So the synchronisation error is $\frac{v\,x_m}{c^2}$ *slow* in moving system units and $\gamma\frac{v\,x_m}{c^2}$ *slow* in stationary system units. Note that since the integration is of the velocity of the clock in the moving system, the result is x_m.

Light pulse synchronisation

If at the moment the two origins are coincident, a light pulse is emitted from the origin of the stationary system and travels to a point (x', y', z') in the moving system in a time t as measured in the stationary system, then it travels a distance given by:

$$d^2 = \left(v\,t + \frac{x'}{\gamma}\right)^2 + y'^2 + z'^2$$

Note that we have to change the length x into stationary system units by dividing by γ. This distance is equal to ct in the stationary system, so we can equate the two to form an equation in t and solve it.

$$c^2t^2 = \left(v\,t + \sqrt{1 - \frac{v^2}{c^2}}\,x'\right)^2 + y'^2 + z'^2$$

This is a standard solution: we expand to get a quadratic in t and solve by the formula

$$t = -\gamma\frac{v\,x'}{c^2} + \gamma\frac{\sqrt{x'^2 + y'^2 + z'^2}}{c}$$

Now $\sqrt{x'^2 + y'^2 + z'^2}$ is just the distance it is seen to travel in the moving system, so dividing by c gives the time t' in moving system units and the factor γ turns the time into stationary system units giving:

$$t = \gamma\left(t' - \frac{v\,x'}{c^2}\right)$$

Thus there is a synchronisation error of $-\gamma\frac{v\,x'}{c^2}$ in stationary system units and $-\frac{v\,x'}{c^2}$ in moving system units. This agrees with the error from moving a clock. It is deliberately written differently to make the reader think about the result.