The mass increase

HOME: The Physics of Bruce Harvey

Introduction

The reader will find this section easier to understand if they have previously read the Electromagnetism section Electromagnetic Mass

Matter essentially consists of electric charges. Mass is not something matter is made of, but a property matter appears to posses by virtue of the fact that moving charges generate magnetic fields which contain their kinetic energy. But a moving magnetic field generates an electric field which acts on the charge. At near light speed, this becomes significant causing its surface and electric flux to contract in the direction of motion. This "Lorentz contraction" increases the flux density and energy content of the magnetic field producing the increase in inertial mass.

In the high energy lab, due to the relative magnitudes of the speed of the earth through the background and the speed of high energy particles, we can say that to all intent and purpose, we are effectively part of the stationary system observing the moving system of the particle. The inertial mass of the high energy particles appears to be increased by factors of γ when resisting centripetal acceleration and γ^3 when resisting linear acceleration.

But in the teaching lab, we are unable to detect the effects of our motion through the background and the null results of our experiments is our main source of information. From this we infer that our rulers have contracted in the direction of motion and our clocks slowed. If all time dependant processes are to be slowed by a factor of γ, then for us in the teaching laboratory, the inference we draw from our null results is that inertial mass appears to be increased by factors of γ^2 when resisting centripetal acceleration and γ^4 when resisting linear acceleration. We show later how the effects on our rulers and clocks results in changes in the real magnitude of our units. In particular, the slowing of clocks affects our measurement of frequency ν and hence the energy $E = h\nu$ of a photon. However energy and mass are equivalent, so the reduced magnitude of the unit of energy equates to a reduced unit of mass. Thus the mass measured in stationary system units is increased by factors of γ when resisting centripetal acceleration and γ^3 and there is no inconsistency.

Historical note

The years around the turn of the 19th and 20th centuries saw some of the most exciting developments in physics as the structure of the atom began to be revealed. Central to this was the discovery of the electron and the hypothesis that the whole of the atom could be explained in terms of electron like components. Beta radiation was shown to consist of electrons travelling at near light speed. Experiments to measure the charge to mass ratio of an electron derived its speed as a necessary part of the calculation. The increases in resistance to linear and centripetal acceleration were revealed and the electron was said to possess two kinds of mass. Longitudinal mass m_l resisted linear acceleration and transverse mass m_t resisted centripetal acceleration. But the accuracy of these results was not great enough to differentiate between the various theoretical models. Both Abraham and Lorentz produced derivations based on different assumptions about how the contraction in length affected the electron's surface and fields. Although Lorentz's result $m_t = \gamma m_0$ and $m_l = \gamma^3 m_0$ would be adopted, the early experimental data favoured Abraham's result. In his 1906 lectures (later to be published with notes in 1915 as "The Theory of Electrons"[3i]) Lorentz deferred to Abraham's result, latter adding a footnote to the published lectures to correct this. It is now customary to say that mass is increased by a factor of γ and that a further factor of γ^2 occurs in linear acceleration.

The effects of the Lorentz contraction are quite complex and the way we see the effects depends on whether we are in the stationary system or the moving system. When we do experiments in the teaching lab, we are part of the moving system of our town as the earth spins and orbits the sun, all moving through the background. The GPS satellite clocks give us real evidence of the effect of motion on atomic clocks and we infer from this that all clocks are affected. Most clocks rely on some form of oscillating mechanism which will slow by a factor of γ if within the moving system, $m_t = \gamma^2 m_0$ and $m_l = \gamma^4 m_0$. Einstein's paper of 1905 managed to derive both results saying that it all depended on one's interpretation of mass! The three derivations differ in that Lorentz attributes the increase in mass as being due to an increase in the energy content of the magnetic field while Abraham also includes the increase in the energy content of the electric field, but Einstein only considers the increase in energy in the electric field dismissing the magnetic field as an artefact of observation.

The author's first attempts were made in ignorance of the earlier work. The first gave a non-relativistic derivation of the force resisting linear acceleration attempting to explain the generation of the force as one of electromagnetic induction. The second attempt succeed in explaining the centrifugal force as well and the third attempt took into account the relativistic effects and could be considered as a correction of Lorentz's theory. One of the author's achievements was the realization that the contraction applied to the flux density $\vec{D}$ and potential ϕ descriptors of the electric field leading to an explanation of the invariance of the energy content of the electric field.

In our unified theory, there is no room for fudges or magic incantations, everything has to be explained by simple causal processes of nature. The development of the author's theory of the quantised energy levels of hydrogen revealed an inconsistency in that any increase in the inertial mass of its electrons should result in an increase in the magnitude of the quantised energy levels of the atom. The author has repeatedly revised this section in the previous months, finding what appeared to be the answer, but then doubting the analysis when proof reading a few weeks later. There just had to be an answer. When it finally emerged, the answer was so simple. In our moving system, the unit of energy is reduced in magnitude by a factor γ. The atomic clock measures the energy levels within the atom though the relationship $E = h\nu$. Since the atomic clock has slowed by a factor of γ the frequency ν is reduced. Planck's constant $h = 2e\,\Phi_0$ must be invariant because it depends on the fundamental quanta of electric charge and magnetic flux. Therefore the energy levels are reduced. The equations which determine the quantised energy levels of the atom contain the permittivity of space ε_0. If it remains constant, then the equations predict an increase in energy levels, but it is subject to change explaining both the reduced energy levels and the reduction in the magnitude of the unit of energy.

This insight solved the problem of whether inertial mass is increased by factors of γ^2 and γ^4 or by factors of γ and γ^3. Mass and energy are equivalent, so if the unit of energy had decreased in size due to the contraction, then so has the unit of mass. The stationary system unit of mass is γ times the moving system unit of mass, so we must divide the factors of γ^2 and γ^4 by γ to give the factors γ and γ^3 observed from the stationary system.

If we go back to Lorentz's derivation, we see that he worked using moving system co-ordinates and obtained the raw result of γ^2 and γ^4, but then argued that as the volume element was reduced in size, he must divide by γ to give γ and γ^3. This argument is incorrect because his choice of co-ordinates has changed the integrals into parametric integrals and they give the correct answer.

Lorentz simply ignored any possible effect on the energy content of the electric field, but Poincaré both pointed this out and provided an explanation saying that the increase was balanced by a decrease in the internal stress energy of electron. Einstein, being his own man ignored the magnetic field and identified the increase in electric energy as the kinetic energy while insisting that they were only artefacts of observation.

The author realised that the electric field of the electron is invariant attributing this to two factors. The contraction increases the y and z components of the electric flux density $\vec{D}$ and increases the x component of the electric field strength $\vec{E}$ which is equal to the gradient of the potential $\nabla\phi$. But was then in error in accepting Lorentz's conclusion that the reduced capacity of the volume element $d\tau$ results in a decrease in the result of the integral. The volume element $\delta\tau$ remains the same volume element containing the same chunk of flux and what were co-ordinates in Euclidean space describing its position and size have now become parameters. This turns the integral into a parametric integral summing all such volume elements.

$$\vec{D}' = \begin{pmatrix} D_x \\ \gamma D_y \\ \gamma D_z \end{pmatrix} \qquad \vec{E}' = \nabla\phi' = \begin{pmatrix} \gamma \frac{d\phi}{dx} \\ \frac{d\phi}{dy} \\ \frac{d\phi}{dz} \end{pmatrix}$$

$$\text{And} \qquad \mathcal{E}_e' = \int \frac{1}{2}\vec{D}' \cdot \vec{E}' \, d\tau = \int \frac{1}{2}\begin{pmatrix} D_x \\ \gamma D_y \\ \gamma D_z \end{pmatrix} \cdot \begin{pmatrix} \gamma \frac{d\phi}{dx} \\ \frac{d\phi}{dy} \\ \frac{d\phi}{dz} \end{pmatrix} d\tau = \int \frac{1}{2}\gamma \vec{D} \cdot \vec{E} \, d\tau$$

However, one of the effects of the contraction is to reduce the magnitude of the units of energy within the moving system by a factor of γ. So the answer above is in moving system units and must be divided by γ to give $\frac{1}{2}\int \vec{D} \cdot \vec{E} \, d\tau$ stationary system units showing that the energy content is constant.

Electromagnetic mass

The theory of electromagnetic mass is based on the assumption that the kinetic energy $\mathcal{E}_m = \frac{1}{2}m v^2$ of an electron is stored in the magnetic field generated by its motion. At normal speeds the kinetic energy $\mathcal{E}_m = \frac{1}{2}m v^2$ of an electron stored in its magnetic field can be calculated thus:

$$\vec{B} = \mu_0 \vec{v} \wedge \vec{D} = \frac{\mu_0 q}{4\pi r^2} \vec{v} \wedge \hat{r}$$

$$\mathcal{E}_m = \int_{volume} \frac{1}{2\mu_0} B^2 = \int \frac{\mu_0}{2} \left(\vec{u} \wedge \vec{D}\right)^2 d\tau = \int_0^{2\pi} \int_0^{\pi} \int_a^{\infty} \frac{\mu_0 q^2 u^2 \sin^2\theta}{32\pi^2 r^4} r^2 \sin\theta \, dr \, d\theta \, d\varphi = \frac{\mu_0 q^2 u^2}{12\pi a}$$

$$\text{Equating} \qquad \mathcal{E}_m = \frac{\mu_0 q^2}{12\pi a} v^2 \qquad \text{with} \qquad KE = \frac{1}{2} m v^2 \qquad \Rightarrow \quad a = \frac{\mu_0 q^2}{6\pi m}$$

where a is the radius of the electron. [Like Lorentz we use electron as a generic term for any elementary charged particle. Hence the use of q rather than $-e$ for the charge.]

We boldly stated that Lorentz's use of the auxiliary co-ordinates simply changed the integrals into parametric integrals and they yielded the correct result without the need to divide by γ. Perhaps we should take the effort to do the integration in Euclidean co-ordinates for the energy content of the magnetic field and show that this is so. As the speed increases and the Lorentz contraction becomes significant. It increases the flux density of the magnetic field by a factor γ and also affects the values of θ, $d\theta$, r and dr as measured against Euclidean space.

$$\theta \to \tan^{-1}\left(\gamma \tan\theta\right) \qquad d\theta \to \gamma \frac{1 + \tan^2\theta}{1 + \gamma^2 \tan^2\theta} d\theta \qquad \sin\theta \to \gamma \frac{\tan\theta}{\sqrt{1 + \gamma^2 \tan^2\theta}}$$

$$r \to r\sqrt{\frac{\cos^2\theta}{\gamma^2} + (1 - \cos^2\theta)} \qquad dr \to \sqrt{\frac{\cos^2\theta}{\gamma^2} + (1 - \cos^2\theta)} \, dr$$

The limit a becomes $\frac{a}{\sqrt{\frac{cos^2\theta}{\gamma^2} + (1 - cos^2\theta)}}$ as can be seen in the example:

$$\int_a^\infty \frac{1}{r^2}\,dr \;=\; \frac{1}{a} \qquad \rightarrow \qquad \int_{\frac{a}{k}}^\infty \frac{1}{(k\,r)^2}\,k\,dr \;=\; \frac{1}{a}$$

When these new values are substituted into the integral, the function tan^{-1} is only well defined from $-\pi/2$ to $\pi/2$, so we need to integrate over θ for half the interval and double the result. We must also change the order of integration because r is now a function of θ. The integral becomes:

$$2\int_0^{2\pi}\int_{\frac{a}{\sqrt{\frac{cos^2\theta}{\gamma^2} + (1 - cos^2\theta)}}}^\infty \int_0^{\frac{\pi}{2}} \gamma^2 \frac{\mu_0\,q^2\,u^2\,\left(\gamma\frac{tan\theta}{\sqrt{1+\gamma^2\,tan^2\theta}}\right)^3}{32\pi^2\left(\frac{cos^2\theta}{\gamma^2} + \left(1 - cos^2\theta\right)\right)r^2}\sqrt{\frac{cos^2\theta}{\gamma^2} + \left(1 - cos^2\theta\right)}\;\gamma\frac{1 + tan^2\theta}{1 + \gamma^2\,tan^2\theta}\,d\theta\,dr\,d\varphi$$

Fortunately MathCad makes light work of this to get:

$$\mathcal{E} \;=\; \gamma^2\,\frac{\mu_0\,q^2\,u^2}{12\pi\,a}$$

A result which is produced more simply by the parametric integration:

$$\mathcal{E}_m{}' \;=\; \int_{volume}\frac{1}{2\mu_0}\,(\gamma B)^2\,J \;=\; \int\gamma^2\,\frac{\mu_0}{2}\,\left(\vec{u}\,\wedge\,\vec{D}\right)^2\,d\tau$$
$$=\int_0^{2\pi}\int_0^{\pi}\int_a^\infty \frac{\gamma^2\,\mu_0\,q^2\,u^2\,sin^2\theta}{32\pi^2\,r^4}\,r^2\,sin\,\theta\,dr\,d\theta\,d\varphi \;=\; \gamma^2\,\frac{\mu_0\,q^2\,u^2}{12\pi\,a}$$

But the effect of the Lorentz contraction has also reduced the magnitude of the unit of energy by a factor of γ so that the energy content $\gamma^2\,\frac{\mu_0\,q^2\,u^2}{12\pi\,a}$ is measured in moving system units and must be divided by γ to express it in stationary system units.

It must be understood that the kinetic energy is actually stored in the moving system of the electron's magnetic field. When we accelerate a particle in the high energy lab, we are doing work to increase its kinetic energy over the range of its increasing velocity and this breaks the strict one to one relationship between the two measures of kinetic energy. Performing the integrals in the stationary system $KE \;=\; (\gamma\;-\;1)\,m\,c^2$ which is not equal to $\frac{1}{2}\gamma\,m\,v^2$. However, because of the problems calculators and computers have with loss of accuracy, the latter is more accurate for velocities up to 150,000 m/s when using Mathcad. For near light speed, $(\gamma\;-\;1)\,m\,c^2\;>\;\frac{1}{2}\gamma\,m\,v^2$, but only reaches twice the magnitude when $\gamma\;\approx\;7{,}000$. The two answers diverge because as the electron is accelerated, and kinetic energy is added, the units in which it is measured within the stationary system are γ times the magnitude of those of the moving system, but γ is constantly increasing. A good analogy would be a Norwegian grandfather who has two grand children, one living in France and the other in England and paid 1000 Nor. Kr. into their bank accounts every year. As exchange rates fluctuate they will always be arguing over who got more.

We diverge considerably from Lorentz's simple derivation which does not give an adequate explanation of centrifugal force. Lorentz deduced the relationship between transverse and longitudinal mass. He derived the longitudinal mass from an integration of the total energy content of the magnetic field. This is very much simpler than the approach we have followed. In order to unmask the process by which centrifugal force is generated, we have to consider the rate of change in energy content of the magnetic field. Now in centripetal acceleration there is no net change, only a rotation of the magnetic field as the direction of motion changes (the flux being orthogonal to the direction of motion). The only way to obtain the correct result is to assume that changes in the magnetic field must be accommodated by the movement of energy parallel to the electric field. Nature does not allow the electric field to rotate, so we may construct a volume element outwards from a surface element parallel to $\vec{D}$ and consider the energy content of the magnetic field within it. This gives us a rate of change of energy within the volume element which we equate to a force on the surface element. The centrifugal force exists as the sum of these forces over the surface elements.

Before we can proceed with the derivation, we need to quote two identities of which the second one is not obvious and from the algebraic perspective, it is very unusual:

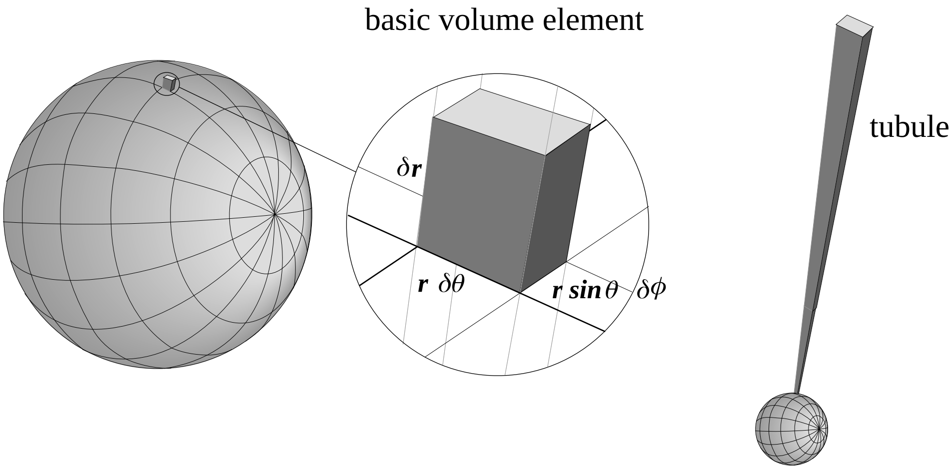
$$\frac{d}{dt}\gamma^n = \frac{n\,v}{c^2}\gamma^{n+2}\frac{dv}{dt} \qquad \text{and} \qquad \frac{v^2}{c^2}\gamma^2 + 1 = \gamma^2$$

We will also need to understand that for functions containing scalars and vectors, the normal rules for differentiating compound functions apply so long as the type and order of multiplication are preserved. Finally we use a technique in which the quadruple scalar product $\vec{A} \wedge \vec{B} \cdot \vec{C} \wedge \vec{D}$ may be treated as a triple scalar product and subjected to cyclic rotation, for example:

$$\vec{A} \wedge \vec{B} \cdot \vec{C} \wedge \vec{D} = \vec{A} \wedge \vec{B} \cdot \left(\vec{C} \wedge \vec{D}\right) = \vec{B} \wedge \left(\vec{C} \wedge \vec{D}\right) \cdot \vec{A}$$

The following derivation is simple when one knows how to do it, but the fact that all of the above need to be used make its discovery almost impossible.

According to classical theory, the moving charge is surrounded by a magnetic field $\vec{B} = \frac{\mu_0\, q}{4\,\pi\, r^2}\vec{v} \wedge \hat{r}$. The Lorentz contraction increases the electric flux density $\vec{D} = \frac{q}{4\,\pi\, r^2}\hat{r}$ by a factor of $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ increasing the magnetic flux density to $\vec{B} = \frac{\gamma\,\mu_0\, q}{4\,\pi\, r^2}\vec{v} \wedge \hat{r}$. The magnetic field has an energy density $\frac{1}{2\,\mu_0}B^2 = \frac{\gamma^2\,\mu_0\, q^2}{32\,\pi^2\, r^4}(\vec{v} \wedge \hat{r})^2$. We consider a surface element of area $\delta A = r^2\,\delta\omega = r^2 \sin\theta\,\delta\theta\,\delta\phi$ and the conical volume element which can be constructed outwards from the surface element everywhere parallel to the electric field. Such volume elements will herein after be referred to as "tubules" since they are constructed according to the rule devised by Faraday to define what latter became known as "Faraday tubes".



We work in Lorentz's auxiliary co-ordinates which move with the electron and suffer a contraction in length. From the discussion above, we know that they have become parameters describing the geometry of the field and giving the energy density when account is taken of the effect of the contraction on energy density. The basic volume element of the tubule is $\delta\tau = r^2\,\delta\omega\,\delta r$ and the energy content of the magnetic field within the tubule is:

$$\delta\mathscr{E}_m = \frac{\gamma^2 \mu_0 q^2}{32\,\pi^2} (\vec{v} \wedge \hat{r})^2 \int_{r_0}^{\infty} \frac{1}{r^4} r^2 \delta\omega\, dr = \frac{\gamma^2 \mu_0 q^2}{32\,\pi^2 r_0} (\vec{v} \wedge \hat{r})^2 \delta\omega$$

We can now differentiate this with respect to time to find the rate of change of energy content of the magnetic field within the tubule. We must remember that γ is a function of velocity. Everything which remains constant with time can be left outside the differentiation.

$$\frac{d}{dt}\delta\mathscr{E}_m = \frac{d}{dt}\left(\frac{\gamma^2 \mu_0 q^2}{32\,\pi^2 r_0} (\vec{v} \wedge \hat{r})^2 \delta\omega \right) = \frac{\mu_0 q^2}{32\,\pi^2 r_0} \frac{d}{dt}\left(\gamma^2 (\vec{v} \wedge \hat{r})^2\right) \delta\omega$$

$$\frac{d}{dt}\gamma^2 (\vec{v} \wedge \hat{r})^2 = \frac{d}{dt}\left(\gamma^2\right)(\vec{v} \wedge \hat{r})^2 + \gamma^2 \frac{d}{dt}\left((\vec{v} \wedge \hat{r})^2\right)$$

$$= \frac{2v}{c^2}\gamma^4 \frac{dv}{dt} (\vec{v} \wedge \hat{r})^2 + 2\gamma^2 (\vec{v} \wedge \hat{r}) \cdot \left(\frac{d\vec{v}}{dt} \wedge \hat{r}\right)$$

We write $\frac{d\vec{v}}{dt} = \vec{a}$ as $a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$ and note that at this instant $\hat{v} = \hat{i}$ and $\vec{v} = v\,\hat{i}$. Then $\frac{dv}{dt} = a_x$ since it measures the rate of change in magnitude in $\vec{v}$.

$$\frac{d}{dt}\gamma^2 (\vec{v} \wedge \hat{r})^2 = 2\gamma^2 \left(a_x \frac{v}{c^2}\gamma^2 \vec{v} \wedge \hat{r} + \left(a_x \hat{i} + a_y \hat{j} + a_z \hat{k}\right) \wedge \hat{r}\right) \cdot \left(\vec{v} \wedge \hat{r}\right)$$

The next step is to separate the magnitude v and direction $\hat{i}$ of $\vec{v}$ and rearrange $a_x \frac{v}{c^2}\gamma^2\vec{v} \rightarrow \frac{v^2}{c^2}\gamma^2 a_x \hat{i}$. Then collecting the two terms containing a_x and using the γ^2 identity:

$$\frac{d}{dt}\gamma^2 (\vec{v} \wedge \hat{r})^2 = 2\gamma^2 \left(\left(\gamma^2 a_x \hat{i} + a_y \hat{j} + a_z \hat{k}\right) \wedge \hat{r}\right) \cdot \left(\vec{v} \wedge \hat{r}\right)$$

Writing $\vec{a}_\gamma = \gamma^2 a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$ and turning the quadruple scalar product into a triple scalar product:

$$\frac{d}{dt}\gamma^2 (\vec{v} \wedge \hat{r})^2 = 2\gamma^2\ \vec{v} \wedge \hat{r} \cdot \left(\vec{a}_\gamma \wedge \hat{r}\right)$$

$$\frac{d}{dt}\delta\mathscr{E}_m = \frac{d}{dt}\left(\frac{\gamma^2 \mu_0 q^2}{32\,\pi^2 r_0} (\vec{v} \wedge \hat{r})^2 \delta\omega \right) = \frac{\mu_0 q^2}{32\,\pi^2 r_0}\ 2\gamma^2\ \hat{r} \wedge \left(\vec{a}_\gamma \wedge \hat{r}\right) \cdot \vec{v}\, \delta\omega$$

We are now in a position to equate the change in energy with the rate of work done by a force moving with velocity $\vec{v}$ in time δt.

$$\delta\vec{F} \cdot \vec{v}\delta t = \frac{d}{dt}\delta\mathscr{E}_m\, \delta t$$

$$\delta\vec{F} \cdot \vec{v} = \frac{\mu_0 q^2}{32\,\pi^2 r_0}\ 2\gamma^2 \delta\omega\ \hat{r} \wedge \left(\vec{a}_\gamma \wedge \hat{r}\right) \cdot \vec{v}$$

Note that although we chose to impose co-ordinates with the x axis along the direction of motion, this equation requires only that the origin is at the centre of the sphere. Although the dot product does not in general cancel, the fact that this is true for all $\vec{v}$ in this equation implies that:

$$\delta\vec{F} = \frac{\mu_0 q^2}{16\,\pi^2 r_0}\ \gamma^2\ \hat{r} \wedge \left(\vec{a}_\gamma \wedge \hat{r}\right) \delta\omega$$

Let us remind ourselves that we have just found the force on the surface element of solid angle $\delta\omega$. We may now integrate over the area of the sphere to find the force required to produce the centripetal acceleration.

$$\vec{F} \;=\; \frac{\mu_0\, q^2}{16\,\pi^2\, r_0}\gamma^2 \int \hat{r} \wedge \left(\vec{a}_\gamma \wedge \hat{r}\right) d\omega$$

$$\vec{F} \;=\; \frac{\mu_0\, q^2}{16\,\pi^2\, r_0}\,\gamma^2 \int_0^{2\pi}\!\!\int_0^{\pi} \hat{r} \wedge \left(\vec{a}_\gamma \wedge \hat{r}\right) \sin\theta \;\, d\theta \;\, d\phi$$

This calculation is best done in Cartesian co-ordinates expanding the vector product, then integrating. The essentials of this have been captured from a Mathcad file.

$$\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\cdot\cos(\phi)\\ \sin(\theta)\cdot\sin(\phi)\end{pmatrix}\times\left[\begin{pmatrix}\gamma^2 a_x\\ a_y\\ a_z\end{pmatrix}\times\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\cdot\cos(\phi)\\ \sin(\theta)\cdot\sin(\phi)\end{pmatrix}\right] = \begin{pmatrix}-\sin(\theta)\cdot\cos(\phi)\cdot a_y\cdot\cos(\theta) - \sin(\theta)\cdot\sin(\phi)\cdot a_z\cdot\cos(\theta) + \gamma^2\cdot a_x - \gamma^2\cdot a_x\cdot\cos(\theta)^2\\ a_y - a_y\cdot\cos(\phi)^2 + a_y\cdot\cos(\theta)^2\cdot\cos(\phi)^2 - \sin(\phi)\cdot a_z\cdot\cos(\phi) + \sin(\phi)\cdot a_z\cdot\cos(\phi)\cdot\cos(\theta)^2 - \cos(\theta)\cdot\gamma^2\cdot a_x\cdot\sin(\theta)\cdot\cos(\phi)\\ a_z\cdot\cos(\theta)^2 - \cos(\theta)\cdot\gamma^2\cdot a_x\cdot\sin(\theta)\cdot\sin(\phi) - \cos(\phi)\cdot a_y\cdot\sin(\phi) + \cos(\phi)\cdot a_y\cdot\sin(\phi)\cdot\cos(\theta)^2 + \cos(\phi)^2\cdot a_z - \cos(\phi)^2\cdot a_z\cdot\cos(\theta)^2\end{pmatrix}$$

$$\int_0^{2\pi}\!\!\int_0^{\pi}\left(-\sin(\theta)\cdot\cos(\phi)\cdot a_y\cdot\cos(\theta) - \sin(\theta)\cdot\sin(\phi)\cdot a_z\cdot\cos(\theta) + \gamma^2\cdot a_x - \gamma^2\cdot a_x\cdot\cos(\theta)^2\right)\cdot\sin(\theta)\, d\theta\, d\phi \rightarrow \frac{8}{3}\cdot\pi\cdot\gamma^2\cdot a_x$$

$$\int_0^{2\pi}\!\!\int_0^{\pi}\left(a_y - a_y\cdot\cos(\phi)^2 + a_y\cdot\cos(\theta)^2\cdot\cos(\phi)^2 - \sin(\phi)\cdot a_z\cdot\cos(\phi) + \sin(\phi)\cdot a_z\cdot\cos(\phi)\cdot\cos(\theta)^2 - \cos(\theta)\cdot\gamma^2\cdot a_x\cdot\sin(\theta)\cdot\cos(\phi)\right)\cdot\sin(\theta)\, d\theta\, d\phi \rightarrow \frac{8}{3}\cdot a_y\cdot\pi$$

$$\int_0^{2\pi}\!\!\int_0^{\pi}\left(a_z\cdot\cos(\theta)^2 - \cos(\theta)\cdot\gamma^2\cdot a_x\cdot\sin(\theta)\cdot\sin(\phi) - \cos(\phi)\cdot a_y\cdot\sin(\phi) + \cos(\phi)\cdot a_y\cdot\sin(\phi)\cdot\cos(\theta)^2 + \cos(\phi)^2\cdot a_z - \cos(\phi)^2\cdot a_z\cdot\cos(\theta)^2\right)\cdot\sin(\theta)\, d\theta\, d\phi \rightarrow \frac{8}{3}\cdot a_z\cdot\pi$$

$$\int_0^{2\pi}\!\!\int_0^{\pi}\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\cdot\cos(\phi)\\ \sin(\theta)\cdot\sin(\phi)\end{pmatrix}\times\left[\begin{pmatrix}\gamma^2 a_x\\ a_y\\ a_z\end{pmatrix}\times\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\cdot\cos(\phi)\\ \sin(\theta)\cdot\sin(\phi)\end{pmatrix}\right]\cdot\sin(\theta)\, d\theta\, d\phi = \begin{pmatrix}\frac{8}{3}\cdot\pi\cdot\gamma^2\cdot a_x\\ \frac{8}{3}\pi\cdot a_y\\ \frac{8}{3}\pi\cdot a_z\end{pmatrix}$$

[This may be viewed at up to 400% in Acrobat and will print legibly]

Writing the result of the integration as $\frac{8\pi}{3}\,\vec{a}_\gamma$, the force required to accelerate the electron is:

$$\vec{F} \;=\; \frac{\mu_0\, q^2}{16\,\pi^2\, r_0}\,\gamma^2\,\frac{8\pi}{3}\,\vec{a}_\gamma \;\;=\; \frac{\mu_0\, q^2}{6\,\pi\, r_0}\,\gamma^2\;\vec{a}_\gamma$$

Defining a quantity $m_0 \;=\; \dfrac{\mu_0\, q^2}{6\,\pi\, r_0}$ we have:

$$\vec{F} \;=\; \gamma^2\, m_0\, \vec{a}_\gamma$$

This is a relativistic form of Newton's second law.

$$\vec{F} \;=\; \gamma^2\, m_0\, \vec{a}_\gamma \qquad : \;\vec{a}_\gamma \;=\; \begin{pmatrix}\gamma^2\,\vec{a}_x\\ \vec{a}_y\\ \vec{a}_z\end{pmatrix} \qquad \text{(in moving system units)}$$

This coincides with Lorentz's raw result which he then divided by γ to account for the reduced volume element. We have to take into account the reduced magnitude of the unit of energy. We derived the force equation from an energy equation by cancelling the dot product with the velocity. In doing that we transferred the reduction in the unit of energy to the unit of force. There is a neat way of looking at this. If we consider a force exerted by a rope between the stationary and moving systems. The stationary system would winch in Lorentz contracted rope, but if it was the moving system that winched in the rope it would not be contracted. So there is an inherent velocity ratio and with it a mechanical advantage. The inertial force is:

$$\vec{F} = \gamma\, m_0\, \vec{a}_\gamma \qquad : \ \vec{a}_\gamma = \begin{pmatrix} \gamma^2\, \vec{a}_x \\ \vec{a}_y \\ \vec{a}_z \end{pmatrix} \qquad \text{(in stationary system units)}$$

Which is the relativistic form of Newton's Second Law.