Effect of magnetic field

Stern-Gerlach experiment

This is an html file generated from Mathcad.

This is the mathematical analysis of coupling between a hydrogen like atom and a magnetic field as demonstrated by the Stern-Gerlach experiment.

(Mathcad prefers the use of q for the charge on the electron)

Solution of orbits without magnetic field

Given $\qquad \frac{-1\cdot q^2}{4\cdot\pi\cdot\varepsilon_0\cdot r} = \left(-4\cdot m\cdot\pi^2\cdot r^2\cdot\nu^2\right)$ (1) $\qquad \frac{1}{2}\cdot\nu\cdot q\cdot n\cdot\Phi_0 = \frac{1}{2}\cdot 2\cdot m\cdot\pi^2\cdot r^2\cdot\nu^2$ (2)

$$\mathrm{Find}(r,\nu) \rightarrow \begin{pmatrix} 4\cdot n^2\cdot\Phi_0{}^2\cdot\frac{\varepsilon_0}{\pi\cdot m} \\ \frac{1}{32}\cdot\frac{q}{n^3\cdot\Phi_0{}^3}\cdot\frac{m}{\varepsilon_0{}^2} \end{pmatrix}$$

Dividing equation (1) by r, it then equates forces and we add a term for the Bev f o r c e e, w now have the two equations

$$\frac{-q^2}{4\cdot\pi\cdot\varepsilon_0\cdot r^2} = \left(-4\cdot m\cdot\pi^2\cdot r\cdot\nu^2\right) + B\cdot q\cdot 2\pi r\cdot\nu \quad (3) \qquad \frac{1}{2}\cdot\nu\cdot q\cdot n\cdot\Phi_0 = \frac{1}{2}\cdot 2\cdot m\cdot\pi^2\cdot r^2\cdot\nu^2 \quad (2)$$

We cannot use a solve block, but must solve equation (2) for ν, then substitute for ν (nu) in equation (3).

$$\frac{1}{2}\cdot\nu\cdot q\cdot n\cdot\Phi_0 = \frac{1}{2}\cdot 2\cdot m\cdot\pi^2\cdot r^2\cdot\nu^2 \text{ solve},\nu \;\rightarrow \begin{pmatrix} 0 \\ \frac{1}{2}\cdot q\cdot n\cdot\frac{\Phi_0}{m\cdot\pi^2\cdot r^2} \end{pmatrix}$$

$$\frac{-q^2}{4\cdot\pi\cdot\varepsilon_0\cdot r^2} = \left(-4\cdot m\cdot\pi^2\cdot r\cdot\nu^2\right) + B\cdot q\cdot 2\pi r\cdot\nu \text{ substitute},\nu = \frac{1}{2}\cdot q\cdot n\cdot\frac{\Phi_0}{m\cdot\pi^2\cdot r^2} \;\rightarrow \frac{-1}{4}\cdot\frac{q^2}{\pi\cdot\varepsilon_0\cdot r^2} = \frac{-1}{m\cdot\pi^2\cdot r^3}\cdot q^2\cdot n^2\cdot\Phi_0{}^2 + B\cdot\frac{q^2}{\pi\cdot r}\cdot n\cdot\frac{\Phi_0}{m}$$

We now change the variable from r to $c = \frac{1}{r}$

$$\frac{-1}{4}\cdot\frac{q^2}{\pi\cdot\varepsilon_0\cdot r^2} = \frac{-1}{m\cdot\pi^2\cdot r^3}\cdot q^2\cdot n^2\cdot\Phi_0{}^2 + B\cdot\frac{q^2}{\pi\cdot r}\cdot n\cdot\frac{\Phi_0}{m} \text{ substitute}, r = \frac{1}{c} \;\rightarrow \frac{-1}{4}\cdot\frac{q^2}{\pi\cdot\varepsilon_0}\cdot c^2 = \frac{-1}{m\cdot\pi^2}\cdot c^3\cdot q^2\cdot n^2\cdot\Phi_0{}^2 + B\cdot\frac{q^2}{\pi}\cdot c\cdot n\cdot\frac{\Phi_0}{m}$$

and solve for c (we could divide by c and solve the quadratic, but Mathcad can solve cubics)

$$\frac{-1}{4}\cdot\frac{q^2}{\pi\cdot\varepsilon_0}\cdot c^2 = \frac{-1}{m\cdot\pi^2}\cdot c^3\cdot q^2\cdot n^2\cdot\Phi_0{}^2 + B\cdot\frac{q^2}{\pi}\cdot c\cdot n\cdot\frac{\Phi_0}{m} \text{ solve}, c \;\rightarrow \begin{bmatrix} \frac{1}{8\cdot n^2\cdot\Phi_0{}^2\cdot\varepsilon_0}\cdot\left[\pi\cdot m + \left(\pi^2\cdot m^2 + 64\cdot n^3\cdot\Phi_0{}^3\cdot\varepsilon_0{}^2\cdot B\cdot\pi\right)^{\frac{1}{2}}\right] \\ \frac{1}{8\cdot n^2\cdot\Phi_0{}^2\cdot\varepsilon_0}\cdot\left[\pi\cdot m - \left(\pi^2\cdot m^2 + 64\cdot n^3\cdot\Phi_0{}^3\cdot\varepsilon_0{}^2\cdot B\cdot\pi\right)^{\frac{1}{2}}\right] \\ 0 \end{bmatrix}$$

We take the first solution as the only one which will reduce to $c = \frac{1}{r} = \frac{1}{\left(4\cdot n^2\cdot\Phi_0{}^2\cdot\frac{\varepsilon_0}{Z\cdot m\cdot\pi}\right)}$ when B = 0

Taking out a factor of $m\cdot\pi$ the solution is written

$$\frac{1}{8\cdot n^2\cdot\Phi_0{}^2\cdot\varepsilon_0}\cdot\left[m\cdot\pi\cdot\left(1+\sqrt{1+\frac{64\cdot B\cdot n^3\cdot\Phi_0{}^3\cdot\varepsilon_0{}^2\cdot\pi}{m^2\cdot\pi^2}}\right)\right]$$

Expanding the square root as a Binomial distribution and neglecting higher terms

$$\frac{1}{r}=\frac{1}{8\cdot n^2\cdot\Phi_0{}^2\cdot\varepsilon_0}\cdot\left[m\cdot\pi\cdot\left[1+\left(1+\frac{1}{2}\cdot\frac{64\cdot B\cdot n^3\cdot\Phi_0{}^3\cdot\varepsilon_0{}^2\cdot\pi}{m^2\cdot\pi^2}\right)\right]\right]$$

$$\frac{1}{r}=\frac{1}{8\cdot n^2\cdot\Phi_0{}^2\cdot\varepsilon_0}\cdot\left[m\cdot\pi\cdot\left[1+\left(1+32\cdot n^3\cdot\Phi_0{}^3\cdot\varepsilon_0{}^2\cdot\frac{B}{\pi\cdot m^2}\right)\right]\right]$$

$$\frac{1}{r}=\frac{1}{4\cdot m}\cdot\frac{m^2\cdot\pi+16\cdot B\cdot n^3\cdot\Phi_0{}^3\cdot\varepsilon_0{}^2}{n^2\cdot\Phi_0{}^2\cdot\varepsilon_0}$$

$$\frac{1}{r}=\frac{1}{\left(\frac{1}{4\cdot m}\cdot\frac{m^2\cdot\pi+16\cdot B\cdot n^3\cdot\Phi_0{}^3\cdot\varepsilon_0{}^2}{n^2\cdot\Phi_0{}^2\cdot\varepsilon_0}\right)}\rightarrow\frac{1}{r}=4\cdot\frac{m}{m^2\cdot\pi+16\cdot n^3\cdot\Phi_0{}^3\cdot\varepsilon_0{}^2\cdot B}\cdot n^2\cdot\Phi_0{}^2\cdot\varepsilon_0$$

Which we can check by putting B = 0

$$4 \cdot Z \cdot \frac{m}{Z^2 \cdot m^2 \cdot \pi + 16 \cdot B \cdot n^3 \cdot \Phi_0{}^3 \cdot \varepsilon_0{}^2} \cdot n^2 \cdot \Phi_0{}^2 \cdot \varepsilon_0 \text{ substitute}, B = 0 \;\rightarrow\; \frac{4}{Z \cdot m \cdot \pi} \cdot n^2 \cdot \Phi_0{}^2 \cdot \varepsilon_0$$

OK

The energy of the orbiting electron is most easily calculated using Virial Theorem form the potential energy PE = -2 * KE

$$\frac{1}{2} \cdot \frac{q^2}{4\pi \cdot \varepsilon_o} \cdot \left(\frac{1}{4 \cdot m} \cdot \frac{m^2 \cdot \pi + 16 \cdot B \cdot n^3 \cdot \Phi_0{}^3 \cdot \varepsilon_0{}^2}{n^2 \cdot \Phi_0{}^2 \cdot \varepsilon_0} \right) \rightarrow \frac{1}{32} \cdot \frac{q^2}{\pi \cdot \varepsilon_o \cdot m} \cdot \frac{m^2 \cdot \pi + 16 \cdot n^3 \cdot \Phi_0{}^3 \cdot \varepsilon_0{}^2 \cdot B}{n^2 \cdot \Phi_0{}^2 \cdot \varepsilon_0}$$

Which may be written

$$\frac{1}{32} \cdot \frac{q^2}{\varepsilon_o} \cdot \frac{m}{n^2 \cdot \Phi_0{}^2 \cdot \varepsilon_0} + \frac{1}{2} \cdot \frac{q^2}{\pi \cdot \varepsilon_o \cdot m} \cdot n \cdot \Phi_0 \cdot \varepsilon_0 \cdot B$$

We identify the second term as the coupling energy which is equal to the product of the magnetic moment and the flux density.

The magnetic moment is $\frac{1}{2} \cdot \frac{q^2}{\pi \cdot \varepsilon_o \cdot m} \cdot n \cdot \Phi_0 \cdot \varepsilon_0$

which for n = 1 is the Bohr magneton.