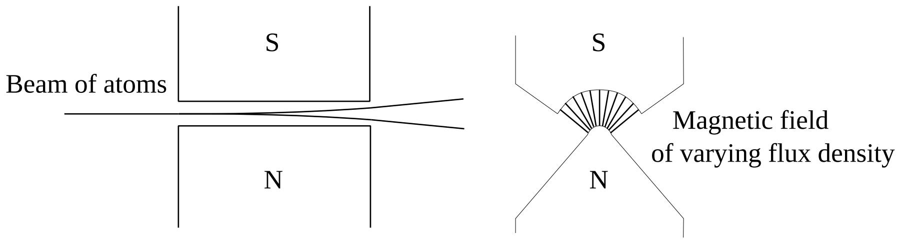
# Stern-Gerlach Experiment

HOME: The Physics of Bruce Harvey

This is the experiment that is said to prove that the electron has an intrinsic magnetic moment. Hydrogen like atoms are projected in a beam through a magnetic field which varies in intensity in a direction perpendicular to the beam. The atoms behave like small magnetic dipoles and are deflected to form two beams depending on whether they align parallel or anti-parallel to the magnetic field. There is an excellent description of the modern version of this experiment which can be downloaded from the apparatus manufacturer PHYWE SYSTEME GMBH. There is another excellent article in Physics Today, December 2003 which explains how the experiment was originally performed to confirm the original Bohr model.




The splitting of the beam into two components confirms that the atoms behave neither as magnetic dipoles, current loops or charged gyroscopes. Magnetic dipoles and current loops should align parallel producing a single deflected beam. Charged gyroscopes should precess maintaining their angle with the field and produce a broadening of the beam. The twofold splitting confirms that there is no gyroscopic action and that the atoms align parallel and anti-parallel with equal probability. We can account for this behaviour with our unified theory.

We can analyse the effect of placing the hydrogen atom in a magnetic field perpendicular to its orbital plane. We restrict ourselves to the case of circular orbits threaded by $n$ quanta of magnetic flux in a background magnetic field of flux density $B$ perpendicular to the orbit. We recall that the quantised energy levels of the hydrogen atom were derived by solving the equations:

$$-\frac{Z e^2}{4\pi\varepsilon_0 r} + 4m\pi^2 r^2 \nu^2 = 0 \qquad (1) \qquad\qquad \frac{1}{2}\nu e n \Phi_0 = \frac{1}{2} 2m\pi^2 r^2 \nu^2 \qquad (2)$$

We modifying the first equation by dividing each side by $r$ to give the equation relating electric force to centrifugal force. Then add a term for the magnetic force $Be\text{v}$. For parallel alignment the $Be\text{v}$ force is outward in the same direction as the centrifugal force attempting to increase flux linkage. The velocity is $\text{v} = 2\pi r \nu$. We then solve the equations:

$$-\frac{Z e^2}{4\pi\varepsilon_0 r^2} + 4m\pi^2 r \nu^2 + B e 2\pi r \nu = 0 \qquad (3)$$

These equations only approximate to the real atom, describing a model in which an electron of mass $m$ orbits a proton of infinite mass.

Solving equation (2) for $\nu$ gives:

$$\nu = \frac{n e \Phi_0}{2\pi^2 m r^2}$$

This is sufficient to determine the quantised behaviour of the electron orbits for when we substitute for $\nu$ in the expressions for angular momentum and magnetic moment, the $r_s$ and $m_s$ cancel:

$$L = 2\pi r^2 \nu m = 2\pi r^2 \frac{n e \Phi_0}{2 \pi^2 m r^2} m = \frac{n e \Phi_0}{\pi} = n \hbar$$

On substitution of $\Phi_0 = \frac{h}{2e}$ the expression reduces to $n \frac{h}{2\pi} = n \hbar$ a multiple of the reduced Planck constant h-bar. Similarly:

$$\mu = I A = e \nu \pi r^2 = e \frac{n e \Phi_0}{2 \pi^2 m r^2} \pi r^2 = \frac{n e^2 \Phi_0}{2\pi m} = \frac{e}{2m} n \hbar$$

To find the energy, we substitute for $\nu$ in equation (3) this into equation (1) and change the variable from radius $r$ to curvature $c = \frac{1}{r}$ giving:

$$-\frac{e^2}{4\pi \varepsilon_0 r^2} + \frac{n^2 e^2 \Phi_0^2}{\pi^2 m r^3} + B \frac{e^2 n \Phi_0}{\pi m r} = 0$$

$$-\frac{e^2}{4\pi \varepsilon_0} c^2 + \frac{n^2 e^2 \Phi_0^2}{\pi^2 m} c^3 + B \frac{e^2 n \Phi_0}{\pi m} c = 0$$

After dividing by $c$, this is a quadratic in $c$ producing two solutions. Selecting the solution which does not reduce to zero when $B \to 0$ and expanding the square root using the binomial theorem we obtain the solutions:

$$\frac{1}{r} = \frac{1}{8 n^2 \Phi_0^2 \varepsilon_0} \left(\pi m + \left(\pi^2 m^2 - 64\pi n^3 \Phi_0^3 \varepsilon_0^2 B\right)^{\frac{1}{2}}\right)$$

$$\frac{1}{r} = \frac{1}{4 n^2 \Phi_0^2 \varepsilon_0} - \frac{4 n \Phi_0 \varepsilon_0 B}{m}$$

From this we can calculate the energy:

$$\mathcal{E} = -\frac{1}{2} \frac{Z e^2}{4 \pi \varepsilon_0 r} = -\frac{m e^2}{32 n^2 \Phi_0^2 \varepsilon_0^2} + \frac{e^2 n \Phi_0}{2 \pi m} B + \ldots$$

There are two terms, the first giving the energy of the atom when $B = 0$ and the second giving the coupling energy. Now the orbiting electron generates a magnetic moment $\mu = \frac{n e^2 \Phi_0}{2 \pi m}$ and the coupling energy is as expected:

$$\Delta\mathcal{E} = \mu B \qquad : \qquad \mu = \frac{n e^2 \Phi_0}{2\pi m} = \frac{n e \hbar}{2m}$$

$$\mathcal{E} = -\frac{m e^2}{32 n^2 \Phi_0^2 \varepsilon_0^2} + \mu B$$

The effect of parallel alignment has been to increase the energy of the orbital system making it less negative. The interesting thing is that the energy contained in the orbital magnetic field $\mathcal{E}_{mag} = \frac{1}{2} \mathcal{E}_{kinetic} = \frac{1}{4} |\mathcal{E}|$ is reduced. The normal effect of parallel alignment of a current loop is to increase the flux threading the loop and so the torque exerted by the magnetic field on the current loop is always directed towards achieving parallel alignment. It does this through a simple motor action in which $2\Delta\mathcal{E}$ is supplied by the circuitry maintaining the current in the loop (and the current of the electromagnet generating the magnetic field) to do $\Delta\mathcal{E}$ of mechanical work and increase the energy content of the flux threading the loop by $\Delta\mathcal{E}$. This action is not possible in the atom for two reasons: the flux threading the orbit is quantised and cannot increase and there is no external circuitry to supply the energy. The result is that there is no preference for parallel alignment. Indeed, the coupling action is quite different from that between a current loop and a magnetic field.

To understand the nature of the coupling, we must take into account the change in H-flux $\mu_0\vec{H}_0 \cdot \vec{A}$ treading the orbit. Here $\vec{H}_0$ is the magnetic intensity of the magnetic field and $\vec{A}$ the area of the orbit expressed as a vector. Electromagnetic theory casually describes the emf in a circuit being equal to the rate at which flux is cutting the circuit. In fact, the physical flux we describe by its flux density $\vec{B}$ never cuts the circuit. What cuts the circuit is the mathematical artefact we describe as magnetic intensity $\vec{H}$ with the result that $\mu_0\left(\vec{H}_0 \cdot \vec{A}\right)$ emerges from (or is adsorbed into) the conductor increasing or decreasing the flux threading the loop. Once we understand this, we realise that although the flux threading the orbit cannot change the emf generated by a change in $\mu_0\left(\vec{H}_0 \cdot \vec{A}\right)$ is still felt. In fact the emf can be explained by the fact that the *Bev* force depends on the velocity of the electrons relative to the H-flux and that as H-flux cuts the circuit, the *Bev* force includes a component in the direction of the conductor so that summing over all the conduction band electrons gives the emf. The coupling action between our atom and the magnetic field is provided by the emf-like component of the *Bev* force doing work to increase the energy of the orbital system.

Our basic assumptions about the relationship between the total, potential, kinetic and orbital magnetic energies are preserved. So the system is still dominated by the orbital mechanics. This leads to a very interesting discovery which could explain how orbits might align anti-parallel to the field. In the normal situation, aligning a current loop (with field $\vec{H}_L$) parallel to a magnetic field $\vec{H}_0$ increases the flux $\Phi = \int_A \mu_0\left(\vec{H}_0 + \vec{H}_L\right) \cdot d\vec{A}$ threading it thereby increasing the energy content of the magnetic field generated by the current loop. But in the atomic system, the coupling energy increases the energy of the system which becomes less negative. The Virial Theorem then tells us that the kinetic energy of the electron is reduced. In our unified theory, the kinetic energy of the electron is shared equally between the orbital magnetic field and the remaining part of the magnetic field which moves with the electron. If we now consider anti-parallel alignment, the coupling energy is negative and the energy of the orbital system becomes more negative increasing the kinetic energy of the electron.

In parallel alignment, the emf-like component of the *Bev* force does $\Delta\mathcal{E}$ work to increase the energy of the orbital system. The orbit moves outwards requiring $2\Delta\mathcal{E}$ of work to be done against the electric force. The extra $\Delta\mathcal{E}$ coming from a reduction in the kinetic energy of the electron half of which is stored in the flux threading the orbit.

In anti-parallel alignment, the emf-like component of the *Bev* force acts in the opposite direction and adsorbs $\Delta\mathcal{E}$ of work to decrease the energy of the orbital system. The orbit moves inwards and the electric potential does $2\Delta\mathcal{E}$ of work. The extra $\Delta\mathcal{E}$ providing the increase in the kinetic energy of the electron half of which is stored in the flux threading the orbit.

Each of these actions is able to flip the orbit, so as the hydrogen like atoms in the beam enter the magnetic field, they are flip to the nearest alignment, either parallel or anti-parallel resulting in the division of the beam in two.

When we are dealing with real atoms, orbital mechanics has another effect. The electron does not orbit the proton. They both orbit about their combined centre of gravity. Mathematics allows us to reduce the problem to an equivalent one in which an electron of reduced mass $\frac{M}{M+m}\,m$ orbits a proton of infinite mass. The Bohr theory uses the reduced mass of the electron, but when we are dealing with a system which is part mechanical, part electrical and part electromagnetic, this introduces small errors.

If $r$ is the distance between the electron and the proton, $m$ the mass of the electron and $M$ the mass of the proton, then the radius of the electron's orbit is $\frac{M}{M+m}\,r$ and the radius of the proton's orbit is $\frac{m}{M+m}\,r$. We come to the same conclusion as Bohr that the quantisation applies to the whole of the angular momentum and this can be derived if the whole of the kinetic energy is equipartitioned. Equations (1) and (3) now become:

$$\frac{1}{2}\nu\, e\, n\, \Phi_0 = \frac{1}{2}\left(2m\,\pi^2\,\left(\tfrac{M}{M+m}r\right)^2\,\nu^2 + M\,\pi^2\,\left(\tfrac{m}{M+m}r\right)^2\,\nu^2\right) \qquad (4)$$

$$-\frac{Z\,e^2}{4\,\pi\,\varepsilon_0\,r^2} + \frac{4\,m\,\pi^2\,\left(\tfrac{M}{M+m}r\right)^2\,\nu^2 + 4\,M\,\pi^2\,\left(\tfrac{m}{M+m}r\right)^2\,\nu^2}{r} + B\,e\,2\pi\left(\tfrac{M}{M+m}r\right)\nu = 0 \qquad (5)$$

In this, $r$ is the distance between the electron and the proton. We can simplify these equations and write then more neatly introduce a constant $\lambda = \frac{M}{M+m}$:

$$\frac{1}{2}\nu\, e\, n\, \Phi_0 = \frac{1}{2}\,2\lambda\, m\,\pi^2 r^2 \nu^2 \qquad (4)$$

$$-\frac{Z\,e^2}{4\,\pi\,\varepsilon_0\,r^2} + 4\lambda\, m\,\pi^2 r\,\nu^2 + B\,e\,2\pi\,r\,\nu = 0 \qquad (5)$$

Solution of equations (4) and (5) follows exactly as above, but with a sprinkling of $\lambda$s to produce the results:

$$L = \frac{n\,e\,\Phi_0}{\pi} = n\,\hbar \qquad \mu = \lambda\frac{n\,e^2\;\Phi_0}{2\pi\,m} = \lambda\frac{e}{2m}n\,\hbar$$

$$\mathcal{E} = -\frac{\lambda\,m\,e^2}{32\,n^2\;\Phi_0^2\,\varepsilon_0^2} + \mu\,B$$

**It is customary for authors at this point to write: "We leave this as an exercise for the student"**

The magnetic moment cannot be measured accurately with the Stern-Gerlach apparatus because the deflection depends on the velocity of the atoms and the measurement of the magnetic field both of which are difficult to accurately determine. The model answers quoted by PHYWE SYSTEME GMBH for their apparatus claims a result within 2.5% of the accepted value. More accurate determination is made by using two sets of Stern-Gerlach magnets. The first splits the beam and allows atoms of either parallel or anti-parallel alignment to be selected. That beam is then passed through a radio frequency magnetic field which can be tuned to flip the atoms. A second set of Stern-Gerlach magnets then analyses the beam.

The frequency depends on the strength of the magnetic field and was originally understood as the frequency with which the orbit would precess if the system behaved as a charged gyroscope. The frequency is called the Larmor frequency after its discoverer. This understanding is hard to reconcile with the probability distribution model of quantum mechanics, but an alternative explanation is that photons of the Larmor frequency have the correct energy to flip the atom from one state to the other.

The theory we have advanced so far indicates two stable states of the orbit, one aligned parallel to the magnetic field and the other anti-parallel. The difference in energy between these two states is:

$$\Delta\mathcal{E} = 2\,B\,\lambda\frac{n\,e^2\;\Phi_0}{2\pi\,m}$$

For hydrogen atoms in their ground state, $n = 1$. We would expect the flip from anti-parallel to parallel which requires an increase in energy to be associated with the adsorption of a photon of frequency of:

$$\nu = \frac{\Delta\mathcal{E}}{h} = \frac{\Delta E}{2\,e\,\Phi_0} = B\,\frac{e}{2\pi\,m}$$

Comparing this with the original interpretation, the Larmor angular velocity is $\omega_0 = B\,\frac{e}{2\,m}$ corresponding to a frequency:

$$\nu_0 = \frac{\omega_0}{2\pi} = B\frac{e}{4\,\pi\,m}$$

We must conclude that the flip corresponds to the adsorption or emission of 2 photons of frequency $\nu_0$.