The Hydrogen Atom

HOME: The Physics of Bruce Harvey

Introduction

A theory has to explain:

- The quantised energy levels revealed by the spectroscopy.
- The Zeeman effect.
- The twofold splitting of the particle beam in the Stern-Gerlach experiment.
- Flipping of the magnetic moment in magnetic resonance.
- The absorption and emission of photons.

Combining electromagnetic theory, the quantisation of magnetic flux and the Virial Theorem of orbital mechanics, the quantised energy levels are predicted.

The Zeeman effect is the splitting of atomic spectral lines when the atom is in a magnetic field. Lorentz proposed that electron orbits coupled with the magnetic field. This could not explain why the atoms should line up anti-parallel as well as parallel to the field or why some lines were spit into more than two. The problem lay in applying the electromagnetic theory of circuits to an electron orbit.

We extend our analysis to include a background magnetic field and discover that orbital mechanics play a vital role. The atom does not behave as we might expect and we find that it may align either parallel or anti-parallel explaining the twofold splitting of the the particle beam in the Stern-Gerlach experiment. The energy difference between the two states is a multiple of the energy of a photon with the Larmor frequency and we predict the existence of quasi stable states allowing for the orbit to be flipped in magnetic resonance experiments. The normal Zeeman effect is obviously due to the twofold alignment and the anomalous Zeeman effect may well be explained by the quasi stable states, but we have not completed the analysis of this.

With regard to the absorption and emission of photons, our theory is less complete. Our analysis of the photon indicates that while its width is about equal to its wavelength, the hole in the centre of its magnetic flux is atom sized. On the other hand, centrifugal force acting on the electron consists of two components, one generated by the part of the magnetic field which moves with it and the other by the stable orbital magnetic field. We think that the addition or loss of a quanta from the orbital field results in the instability which drives the process of adsorption or emission, but it remains to be seen whether or not these elements can be woven together into a full mathematical analysis.

Allowed orbits

The hydrogen atom consists of an electron orbiting a proton. The electron orbit forms a current loop which generates a magnetic field. Magnetic flux is quantised and the magnetic field generated by the current loop must contain an integer number of quanta of magnetic flux.

The motion of the electron is governed by four main factors:

a) Orbital mechanics as expressed by the Virial Theorem.
b) The energy contained in magnetic flux Φ threaded by a current I is $\frac{1}{2} I \Phi$.
c) Magnetic flux is quantised.
d) The kinetic energy of the electron obeys equipartitioning of energy and is equally shared between a

stable orbital field and the remaining core of the magnetic field which continues to move with it.

These factors combine to produce the allowed orbits. In our initial analysis, we will assume that the electron orbits the proton. In fact, both the electron and the proton orbit about the centre of gravity of the hydrogen atom. However, it is customary in orbital mechanics to analysis an equivalent system in which a reduced mass of $m_r = m \frac{M}{M+m}$ orbits an infinite mass. The distance between the objects and the orbital frequency are unaltered.

The last of these, factor (d) is debatable. The principle is correct and when used in all innocence with the reduced mass alternative system, gives the correct result. However when the full analysis, with the electron and proton both orbiting about their centre of gravity, is carried out, it would appear that the stable orbital field contains half of the kinetic energy of the whole orbital system. We will continue with the analysis of the reduced mass model and return to this latter.

The Virial Theorem states that the average energy of the system is half of the average potential energy. Both these quantities are negative and their difference is the positive average kinetic energy which is equal in magnitude to the negative average total energy. The kinetic energy of a free electron is stored in the surrounding magnetic field generated by its motion. We make the assumption that the kinetic energy of an orbital electron is equally partitioned between the stable magnetic field generated by the current loop and the remaining part of the magnetic field which surrounds the electron and travels with it. This leads to the energy equations:

$$E_{potential} = -2\,E_{kinetic} \qquad E_{obital\ magnetic} = \frac{1}{2} E_{kinetic}$$

$$\text{where} \qquad E_{potential} = -\frac{e^2}{4\,\pi\,\varepsilon_0\,r} \qquad E_{kinetic} = \frac{1}{2} m\,(2\,\pi\,r\,\nu)^2$$

$$E_{obital\ magnetic} = \frac{1}{2} I\,\Phi = \frac{1}{2}\,(\nu\,e)(n\,\Phi_0)$$

Where r is the radius of the orbit, e the charge on the electron, ε_0 the permittivity of space, m the mass of the electron, ν ("nu") the orbital frequency and n the number of quanta Φ_0 in the magnetic field. We follow the usual practice of replacing the real orbital system with a reduced mass orbiting an infinite mass and understand m in this sense. The resulting two equations:

$$-\frac{Z\,e^2}{4\,\pi\,\varepsilon_0\,r} = -4\,m_r\,\pi^2 r^2 \nu^2 \qquad (1)$$

$$\frac{1}{2}\,\nu\,e\,n\,\Phi_0 = \frac{1}{2}\,2 m_r\,\pi^2 r^2 \nu^2 \qquad (2)$$

may be solved to give:

$$r = \frac{4\,n^2 \Phi_0^2\,\varepsilon_0}{\pi\,m_r} \qquad \nu = \frac{e\,m_r}{32\,n^3\,\Phi_0^3\,\varepsilon_0^2}$$

Neither of these is directly observable, but we can observe the energy level and the magnetic moment. Multiples of the energy appear on each side of each of our two equations, the simplest form being $-\nu\,e\,n\,\Phi_0$

$$\mathcal{E} = -\nu\,e\,n\,\Phi_0 = -\frac{e\,m_r}{32\,n^3\,\Phi_0^3\,\varepsilon_0^2}\,e\,n\,\Phi_0 = -\frac{m_r\,e^2}{32\,n^2\,\Phi_0^2\,\varepsilon_0^2}$$

The magnetic moment μ for a circular orbit is given by:

$$\mu = IA = \nu e \pi r^2 = \frac{e\, m_r}{32\, n^3\, \Phi_0^3\, \varepsilon_0^2}\, e\, \pi \left(\frac{4\, n^2 \Phi_0^2\, \varepsilon_0}{\pi\, m_r}\right)^2 = \frac{n\, e^2\, \Phi_0}{2\pi\, m_r}$$

Angular momentum is amazingly independent of the mass of the electron and is quantised. Indeed, it can be derived from equation (2) by a simple rearangment of the terms.

$$L = \frac{n\, e\, \Phi_0}{\pi}$$

These reduce to the Bohr equations on substitution of $2\, e\, \Phi_0 = h$. The reader should take care to find a text which uses SI units before checking this result. The Bohr model predictions expressed in SI units are:

$$\mathcal{E} = -\frac{m_r\, e^4}{8\, n^2\, h^2\, \varepsilon_0^2} \qquad \mu = \frac{n\, e\, h}{4\pi\, m_r} = \frac{n\, e\, \hbar}{2\, m_r} \qquad L = \frac{n\, h}{2\pi} = n\, \hbar$$

Reduced mass

The above analysis is over simplistic because in reality, the electron and the proton both describe orbits about their combined centre of gravity. The result is that the predicted energy levels are a little too big and do not quite fit the data from spectroscopy. However, this is a standard problem in classical mechanics and it is know that the real system with an electron of mass m_e and a proton of mass m_p may be replaced by a system in which the electron has a "reduced mass" of $m_r = m_e \frac{m_p}{m_p + m_e}$ and the mass of the proton is considered infinite. The results for r and ν are correct when the reduced mass is used in the formulae. The energy of the system then becomes:

$$\mathcal{E} = -\frac{m_r\, e^2}{32\, n^2\, \Phi_0^2\, \varepsilon_0^2} = -\frac{m_r\, e^4}{8\, n^2\, h^2\, \varepsilon_0^2}$$

These energy levels are in good agreement with spectroscopic data.

The magnetic moment of a current loop is equal to the current I multiplied by the area A. The electron orbit forms a current loop, but its actual radius is $r \frac{m_p}{m_p + m_e}$ where $r = \frac{4\, n^2 \Phi_0^2\, \varepsilon_0}{\pi\, m_r}$ giving:

$$r = \frac{m_p}{m_p + m_e}\, \frac{4\, n^2 \Phi_0^2\, \varepsilon_0}{\pi\, m_e \frac{m_p}{m_p + m_e}} = \frac{4\, n^2 \Phi_0^2\, \varepsilon_0}{\pi\, m_e}$$

The magnetic moment is calculated to be:

$$\mu = IA = q\, \nu\, \pi\, r^2 = q\, \frac{e\, m_r}{32\, n^3\, \Phi_0^3\, \varepsilon_0^2}\, \pi \left(\frac{4\, n^2 \Phi_0^2\, \varepsilon_0}{\pi\, m_e}\right)^2 = \frac{n\, e\, h}{2\pi\, m_e}\, \frac{m_r}{m_e} = \frac{m_p}{m_p + m_e}\, \frac{n\, e\, h}{2\pi\, m_e}$$

This is valid as a descriptor of the orbital magnetic field of the atom. It is not valid for determining the the interaction between the atom and a magnetic field because it does not take into account the effect of orbital mechanics.

Fitting in the flux

We have asserted that the current loop formed by the orbiting electron generates a magnetic field containing an integer number of quanta of magnetic flux. Classical physics only gives two results for the magnetic intensity $\vec{H}$ of a current loop, one for the centre of the loop and the other for the far field. To find $\vec{H}$ in the region of the orbit requires numerical integration using the law of Biot-Savart. Working out from the centre and integrating to find the total flux threading the orbit, we have only accounted for a minute fraction of a quanta of flux by the time we start to get close to the orbit. We reach a point where the numerical result for H closely approximates to the classical result for the magnetic field of a long straight wire $H = \frac{I}{2\pi \mathrm{r}}$ where r is now the distance form the orbital path. Taking the integral of flux density over area of cross section within the limits of distance from the path for which $H = \frac{I}{2\pi \mathrm{r}}$ is a good approximation, we find that the flux extends inwards to within about 1.9 Lorentz electron radii of the path. It is therefore appropriate to think of the orbital magnetic field as forming a tunnel around the orbital path. If we allow for some variation in the tunnel diameter under the action of the approaching electron, this would leave room for the magnetic field which moves with the electron to extend 1 radii from the surface giving it an energy content equal to half of the kinetic energy of the electron. This is in good agreement with our assumption that the kinetic energy of the electron as measured within the atom's rest frame is shared with the stable orbital magnetic field.

Calculating the flux density, for hydrogen in its ground state, at the inner wall of the flux tunnel from the orbital current $I = e\,\nu$:

$$B = \frac{\mu_0 I}{2\pi \mathrm{r}} \qquad I = \frac{e^2 m}{32\, n^3 \Phi_0^3 \varepsilon_0^2} \qquad \mathrm{r} = 1.9\frac{\mu_0 e^2}{6\pi m}$$

$$B = 2.36 \times 10^5\, T$$

This is a quarter of a million times stronger than any magnetic field produced by an electromagnet!

Interaction with a magnetic field

We can analyse the effect of placing the hydrogen atom in a magnetic field perpendicular to its orbital plane. We restrict ourselves to the case of circular orbits threaded by n quanta of magnetic flux. The maths is somewhat diabolical, so we only quote results from the Mathcad worksheet.[1]

Let us first write the equations in full taking into account the fact that the electron and proton both orbit the centre of gravity of the atom:

$$-\frac{Z e^2}{4\pi\varepsilon_0 r} = -4\, m_e \pi^2 \left(\frac{m_p}{m_p + m_e} r\right)^2 \nu^2 - 4\, m_p \pi^2 \left(\frac{m_e}{m_p + m_e} r\right)^2 \nu^2$$

$$\frac{1}{2}\nu\, e\, n\, \Phi_0 = \frac{1}{2} 2 m_e \pi^2 \left(\frac{m_p}{m_p + m_e} r\right)^2 \nu^2 + \frac{1}{2} 2 m_p \pi^2 \left(\frac{m_e}{m_p + m_e} r\right)^2 \nu^2$$

where m_e is the mass of the electron and m_p the mass of the proton. We can simplify these and then introduce a factor $\lambda = \frac{m_p}{m_p + m_e}$ to write them as:

$$-\frac{Z e^2}{4\pi\varepsilon_0 r^2} = -4\lambda\, m_e \pi^2 r\, \nu^2 \qquad (1)$$

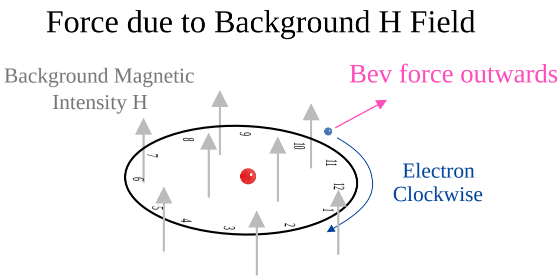
$$\frac{1}{2}\nu\, e\, n\, \Phi_0 = \lambda\, m_e \pi^2 r^2 \nu^2 \qquad (2)$$

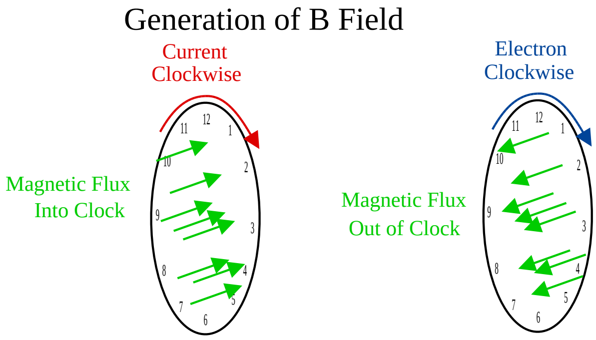
We modify equation (1) by dividing each side by r. This gives the equation relating electric force to centripetal force. [strictly speaking the centrifugal force is $4\, m_e \pi^2\ (\lambda\, r)\ \nu^2$ but the expressions are equal] We need to write this more explicitly as two forces which sum to zero.

1 In fact, Mathcad struggles with this and needs some human intervention to steer it in the right direction.

$$-\frac{Z\,e^2}{4\,\pi\,\varepsilon_0\,r^2} = -4\lambda\,m\,\pi^2 r\,\nu^2 \;\rightarrow\; -\frac{Z\,e^2}{4\,\pi\,\varepsilon_0\,r^2} + 4\lambda\,m\,\pi^2 r\,\nu^2 = 0$$

Now we have the inward electric force plus the outward centrifugal force equal to zero indicating equilibrium. Parallel alignment occurs when the background magnetic intensity $\vec{H}_0$ is in the same direction as the magnetic flux density $\vec{B}_{orbital}$. We just need to make sure we get this right.





We add a term for the magnetic force $Be\text{v}$ on a moving charge. For parallel alignment the $Be\text{v}$ force is outward adding to the centrifugal force as it attempts to increase flux linkage, so the $Be\text{v}$ term is positive. The velocity is $\text{v} = 2\pi\,r\,\nu$. We then solve the equations:

$$-\frac{Z\,e^2}{4\,\pi\,\varepsilon_0\,r^2} + 4m\,\pi^2 r\,\nu^2 + B\,e\,2\pi\,r\,\nu = 0 \qquad \frac{1}{2}\,\nu\,e\,n\,\Phi_0 = m\,\pi^2 r^2 \nu^2$$

Solving the second equation for ν gives:

$$\nu = \frac{n\,e\,\Phi_0}{2\,\pi^2\,m\,r}$$

we then substitute this into the first and change the variable from radius r to curvature $c = \frac{1}{r}$ giving:

$$-\frac{e^2}{4\pi\,\varepsilon_0}\,c^2 + \frac{n^2 e^2 \Phi_0^2}{\pi^2\,m}\,c^3 + B\,\frac{e^2\,n\,\Phi_0}{\pi\,m}\,c = 0$$

After dividing by c, this is a quadratic in c producing two solutions. Selecting the solution which reduces to our previous solution when $B \rightarrow 0$ and expanding the square root using the binomial theorem we obtain the solutions:

$$\frac{1}{r} = \frac{\pi\,m^2 - 16\,B\,n^3\,\Phi_0^3\,\varepsilon_0^2 + \ldots}{4\,m\,n^2\,\Phi_0^2\,\varepsilon_0} = \frac{\pi\,m}{4\,n^2\,\Phi_0^2\,\varepsilon_0} - \frac{4\,B\,n\,\Phi_0\,\varepsilon_0}{m} + \ldots$$

$$\nu = \frac{e\,m}{32\,n^3\,\Phi_0^3\,\varepsilon_0^2} - B\,\frac{e}{\pi\,m} + \ldots$$

The effect of parallel alignment is to cause the radius to increase and the orbital frequency to decrease. From these we can calculate the energy:

$$\mathcal{E} = -\frac{m\,e^2}{32\,n^2\,\Phi_0^2\,\varepsilon_0^2} + B\,\frac{e^2\,n\,\Phi_0}{2\pi\,m} + \ldots$$

There are two terms, the first giving the energy of the atom when $B = 0$ and the second giving the coupling energy. The second term $B\frac{n\,e^2\,\Phi_0}{2\pi\,m}$ is the coupling energy between a magnetic moment $\mu = \frac{n\,e^2\,\Phi_0}{2\pi\,m}$ and a background magnetic field of flux density B in the absence of the atom.

We have now derived a result which appears to be wrong. Every ounce of logic tells us that parallel alignment should result in the radius of the orbit increasing. The Bev force acts outwards: we would expect the orbit to move further away from the nucleus. (The author could not believe it and spent almost a week checking and repeating the calculations.)

However, we would also expect parallel alignment to increase the energy content of the magnetic field. The orbital frequency has increased. The magnetic flux threading the orbit is quantised and remains constant, so the energy content $\frac{1}{2} I\, n\, \Phi_0$ has increased. But an increase in the energy content of the orbital magnetic field corresponds to an increase in the kinetic energy of the electron and the Virial Theorem dictates that the total energy must decrease becoming more negative.

The Virial Theorem and the quantisation of magnetic flux have turned our simple concepts of mechanics and electromagnetism inside out. This should not surprise those readers who ride a bike. It is easy to ride a trike. If one wants to turn to the right, one exerts a torque turning the bars to the right. It is very difficult to ride a bike because it works the opposite way. Exert a torque turning the bars to the right and the bike fights back exerting a greater torque and turning the bars to the left so that the bike turns to the left. Every instinct tells the learner to fight the bike and turn the bars to the right with the result that they fall off. Somehow by accident the brain eventually works out that in order to turn to the right, one has to exert a torque turning the bike to the left and then let the bike win as it tries to turn the bars to the right. So in the equilibrium of a turn to the right, the bars are turned to the right with the rider exerting a torque on the bars to prevent them turning further to the right under the action of the forces generated by the bike.

Of course, the electron does not work on the same principles, but the experience of learning to ride a bike can be used to form the concept that complex systems do not always behave the way we might expect. The Bev force is overpowered by the orbital mechanics and acts to adsorb energy as the orbit contracts.

This leads to a very interesting discovery which could explain how orbits might align anti-parallel to the field. In the normal situation, aligning a current loop ($\vec{H}_{loop}$) parallel to a magnetic field $\vec{H}_0$ increases the flux $\Phi = \int_a \mu_0 \left(\vec{H}_0 + \vec{H}_{loop}\right) \cdot d\vec{A}$ threading it thereby increasing the energy content of the magnetic field generated by the current loop. In fact, almost the opposite happens. The flux threading the orbit is quantised and remains constant, but the energy $\frac{1}{2} I\, n\, \Phi_0$ contained in the flux increases because the orbital frequency and the current which is proportional to it have both increased. This strange behaviour results from the combination of the quantisation of magnetic flux preventing an increase in the flux threading the orbit and the Virial Theorem determining that the energy of the system is negative and equal in magnitude to the kinetic energy.

Suppose that is what happens with an electron orbit. Then if the energy stored in the magnetic flux quanta threading the orbit is increased by $\Delta\mathcal{E}$ the kinetic energy of the electron would be increased by $2\Delta\mathcal{E}$. However, we know that the opposite effect is actually happening. The orbiting electron is subject to the Virial theorem and its kinetic energy is actually decreased. The change in energy of the system is $-B\frac{e^2\, n\, \Phi_0}{2\pi\, m}$. This is contrary to our expectation. Only when the orbit aligns anti-parallel do we get an increase of $B\frac{e^2\, n\, \Phi_0}{2\pi\, m}$ in total energy.

A key point to understand is that force is not a fundamental property of nature. The fundamental property is energy transfer. Force is a property of energy transfer in that force × distance = energy. Forces do not necessarily push or pull things. A force can do one of three things. It can pull/push doing work; it can just maintain a force without doing any work or it can act as a resisting force adsorbing energy as work is done against it. The component of the Bev force parallel to the radii is acting as a resisting force adsorbing energy.

Thus we have two tendencies of nature working against each other. One is the the electromagnetic action which seeks to increase the energy in the magnetic field threading a current loop, the other is the tendency for atoms to decay into their lowest energy state. The result is that each is able to create a stable state. The equal intensity of the split beams in the Stern-Gerlach experiment reveals that the orbit is flipped to whichever is nearest.

The atom does not just happen to be in a magnetic field. Either the field was created in the vicinity of the atom or the atom wondered into the magnetic field. In both cases, there is a change in the H flux threading the orbit. We have shown in our theory of electromagnetism that the *Be*v force does not result from the local magnetic flux density, but from the interaction of the whole of the electric field of the electron with the whole of the magnetic field and that it is only through the integration that we get the very neat result that the force depends on the mathematical artefact $\vec{H}$ at the location of the electron. We should speak of the *μHe*v force, but that is somewhat harder to pronounce.

A more analytical examination of the electron's interaction reveals that the *μHe*v force can be resolved into two components parallel and perpendicular to the radius vector. The perpendicular component is well known to us in electromagnetism where its sum over all the conduction band electrons gives us the EMF which results from a change in the flux threading the circuit. So it is this component of theμHev force which is doing work accelerating the electron as its orbit contracts.

Let us first consider parallel alignment. The component of theBev force in the orbital plan acts outwards increasing the radius of the orbit. Orbital mechanics dictates that the electron slow decreasing its kinetic energy and consequently the current in the current loop. This reduces the energy content of the orbital magnetic field which continues to contain half the kinetic energy of the electron. Although the magnetic energy is decreased, the total energy is that of the orbital system which is negative and equal in magnitude to the kinetic energy. So the energy of the system is increased and becomes less negative. The orbital magnetic field is aligned parallel to the background magnetic field, so the electromagnetic coupling provides the increase in energy of the orbital system.

If we now consider anti parallel alignment. The component of the *Be*v force in the orbital plan acts inwards decreasing the radius of the orbit. Orbital mechanics dictates that the electron speeds up increasing its kinetic energy and the current in the current loop. This increases the energy content of the orbital magnetic field. The

energy of the system is decreased and becomes more negative. The orbital magnetic field is aligned anti-parallel to the background magnetic field, so the electromagnetic coupling provides the decrease in energy of the orbital system. This energy exchange requires the orbit to flip into the anti-parallel position. In this case, if the perpendicular component of theBev force acted as we might expect to act, it would be opposing the required energy transfer. Therefore that component does not exist in its expected form because force is associated with energy transfer. The required energy transfer is negative, so the perpendicular component of the *Be*v force retains its direction, but now acts as a resisting force adsorbing energy.