Advance of the perihelion of Mercury

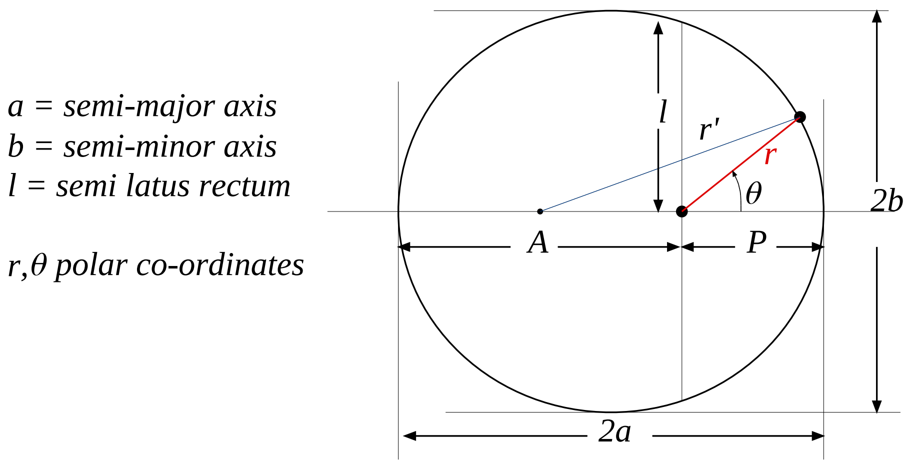
HOME: [The Physics of Bruce Harvey]

In Newton's analysis, the planet knows only its mass, velocity and the force acting on it in the infinitesimal element of time. This causes its velocity to change over time. Unknown to the planet, Newton's mathematics then shows that the planet traces out an ellipse. It must be emphasised that nature knows nothing of the ellipse, only of the local action in the moment of now.

The mathematics of the ellipse is not easy. Some of its properties are beautiful, others are bastards. For instance, its area is $\pi\, a\, b$ (where a and b are the maximum and minimum radii from its centre) but there is no simple expression for the length of its arc. There are a number of ways of drawing a true ellipse, (but draughtsmen often use a quick method with compasses which does not produce a true ellipse). An ellipse has two foci which are the points where you put the pins to draw an ellipse with a loop of string and pencil. The elliptical orbit of a planet has the sun at one of its foci. The nearest distance P to the sun is at the perigee and the longest A at the apogee. The distance when the radius is perpendicular to the major axis is called the semi latus rectum l. We can write an equation in polar co-ordinates for the ellipse:

$$r = \frac{l}{1 + e\cos\theta}$$

where e is the eccentricity.



$$A = \frac{l}{1 - e} \qquad P = \frac{l}{1 + e} \qquad a = \frac{A + P}{2} \qquad b = a\sqrt{1 - e^2} \qquad r + r' = A + P$$

We have discovered that gravitational potential affects the local situation. The action of force on mass and velocity takes place in the local situation. The planet knows only its rate of change of direction. The only local indicator of direction is the equipotential surface of gravitational potential. We are familiar with this concept on earth where we call it the "water level". The equations of motion take place against the "water level". The local rate of change in direction is described by an angular velocity which we shall call $\frac{d\psi}{dt}$. Locally we have clocks and can integrate $\frac{d\psi}{dt}$ over a period of time until we get $\psi = 2\pi$. At this point, the angle between the planet's velocity and the "water level" has returned to its original value and we can say that, from its point of view, the planet has completed an orbit.

It is not possible to describe an elliptical as a nice function of time in the way that we can describe the parabola as $x = a\,t^2,\ y = 2\,a\,t$ or the circle as $x = r\cos(\omega\, t),\ y = r\sin(\omega\, t)$. So Newton's analysis of planetary motion has to derive properties of the orbit and match these to the geometry of the ellipse. This makes the task of authors trying to derive the advance of the perihelion from general relativity very difficult, for their readers will not have the mathematical background to understand the basic method of deriving the equation of the ellipse from the differential equation of motion. Fortunately, this problem does not arise in the theory we present here.

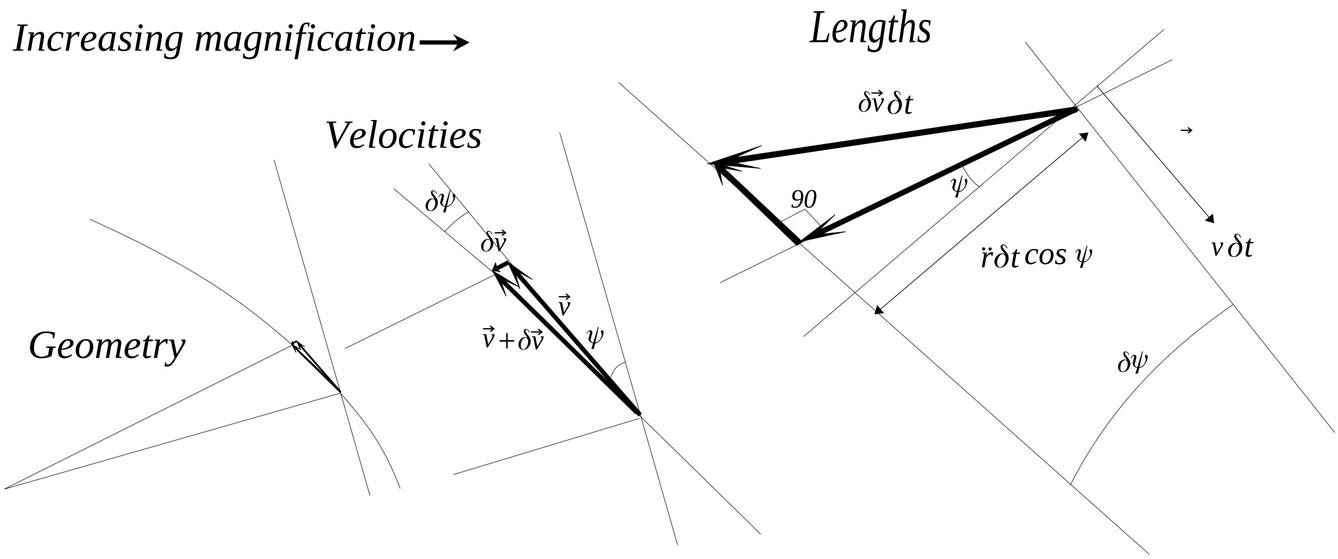
Let us return to our fundamental assertion that there is no such physical entity as space time. It is merely a mathematical artefact describing the numbers we get when we use rulers and clocks to set up co-ordinate systems. The co-ordinate system which we use to describe a planet's orbit are therefore those of Euclidean space and Newtonian time which use units of magnitude as defined in the absence of gravitational potential. The only relevant local effect is that on mass. As we have seen above, inertial mass is increased by a factor $e^{-3\frac{\Phi}{c^2}}$.

The increase in inertial mass has the effect of reducing the radial acceleration. We need to understand the relationship between the radial acceleration $\ddot{r}$ and the rate of rotation $\dot{\psi}$ of the direction. The observed orbit is a supposition of two motions, an elliptical orbit and a very slow rotation. In a perfect elliptical orbit, the direction of the planets path is only perpendicular to the radius from the sun at apogee and perigee. It is most convenient to measure the angle ψ between the direction of the path and the instantaneous line of the normal to the radius, since this lies in the equipotential surface. The affect of the acceleration $\ddot{r}$ is to vary ψ and we can identify an angular velocity $\dot{\psi}$. But as the planet progresses around its orbit, the instantaneous line of the normal to the radius has to be repeatedly redraw. Its direction also has an angular velocity $\dot{\theta}$ and the variation of ψ with time is given by:

$$\psi = \int_0^t \dot{\psi}\, dt - \int_0^t \dot{\theta}\, dt$$

The fact that we do not know the actual functions $\psi(t)$ and $\theta(t)$ does not matter. The advance of the perihelion occurs because gravitational potential effects $\dot{\psi}$ with the result that the time for $\int_0^{T_\psi} \dot{\psi}\, dt = 2\pi$ is greater than the time for $\int_0^{T_\theta} \dot{\theta}\, dt = 2\pi$. Thus, starting at the perigee, when $\psi = 0$; at the next perigee when ψ is again zero, $\theta > 2\pi$ and the perigee has advanced.

The diagram shows the basic geometry at three magnifications. We have superimposed a vector triangle on each, the first two are velocity vectors and the third "journey" vectors.



In time δt, the planet moves $(\vec{v} + \delta\vec{v})\delta t$. The velocity $\delta\vec{v}$ has components parallel and perpendicular to $\vec{r}$. Only the component parallel to $\vec{r}$ affects the change in direction because the other component vanishes in the limit $\delta t \to 0$. The diagram is difficult to draw. In the process of exaggerating angles which tend to zero, the two components no longer actually look perpendicular, The right hand magnification shows the component $\ddot{r}\ \delta t$ of $\delta\vec{v}$ at an angle ψ to the perpendicular to the path, so the critical length is $\ddot{r}\ cos\theta\ \delta t$ and the increment in angle is:

$$\delta\psi\ =\ \frac{\ddot{r}\ cos\theta\ \delta t}{v}\ =\ \frac{\ddot{r}\ r\ \dot{\theta}}{v^2}\ \delta t$$

$$\dot{\psi}\ =\ \frac{\ddot{r}\ r\ \dot{\theta}}{v^2}$$

A planet's orbit is observed from afar noting the changes in its position against the background stars as seen by an observer. The parameters r, v, $\dot{\theta}$ and $\ddot{r}$ are calculated from these observations. Those related to position are the absolute measurements made from afar. The only local action is that of the accelerating force in Newton's law $\vec{F}\ =\ m\,\vec{a}$. We know that the inertial mass is increased by a factor $e^{-3\frac{\Phi}{c^2}}$ which would reduce the acceleration by a factor of $e^{3\frac{\Phi}{c^2}}$, therefore:

$$\text{When } T \text{ satisfies } \int_0^T d\psi\ =\ 2\,\pi\text{ , then } \int_0^T d\theta\ =\ 2\,\pi\,e^{-3\frac{\bar{\Phi}}{c^2}} \qquad \Rightarrow\ T_\psi\ >\ T_\theta$$

The orbital period T is greater from perigee to perigee than position to position. This gives an advance of the perihelion by $2\,\pi\left(e^{-3\frac{\bar{\Phi}}{c^2}}\ -\ 1\right)$ radians per orbit which is 43 seconds of arc per century.

Since the gravitational potential varies over an elliptical orbit, we take a weighted average value of the gravitational potential. We have only used the eccentricity of the orbit to allow us to identify the perigee. The first order effect of gravitational potential is therefore independent of the eccentricity of the orbit. There is a second order effect caused by the fact that the angular momentum $m\,r^2\dot{\theta}$ is a constant. Variations in gravitational potential over the orbit cause small changes in both $\dot{\psi}$ and $\dot{\theta}$. Although these average out over the orbit, the planet spends more time close to the sun because of the effect of gravitational potential on its inertial mass. This will affect the way we calculate the average radius to use in calculating the average gravitational potential. We have not attempted this analysis, but borrow the empirical result first calculated by Gerber and quoted in GR derivations:

$$\frac{24\,\pi^3\,a^2}{T^2\,c^2\,(1\ -\ e^2)}\ \text{radians per orbit}$$

which is more simply expressed in terms of the semi-minor axis as $\frac{24\,\pi^3\,b^2}{T^2\,c^2}$. This gives us the length of the semi-minor axis as the appropriate weighted average for r in calculating the gravitational potential $\Phi\ =\ \frac{GM}{r}$ for substitution into $e^{-3\frac{\Phi}{c^2}}$. [In case the reader is wondering where the 24 comes from: (2 radians per orbit) × (3 in the approximation of $e^{-3\frac{\Phi}{c^2}}$) × (2^2 from converting angular velocity to orbital period)]

Advance per orbit $\qquad 2\,\pi\left(e^{-3\frac{\bar{\Phi}}{c^2}}\ -\ 1\right)\ :\ \bar{\Phi}\ =\ \frac{GM}{b} \qquad b\ =\ \text{semi-minor axis}$