Theory of Everything

Inertial Mass as an Electromagnetic Phenomena

Around the end of the nineteenth century and beginning of the twentieth century some attempts were made to understand matter as a purely electromagnetic phenomena, but the laws of electricity and magnetism were formulated in such a way as to doom the venture to failure. The laws as they have been taught for over a hundred years were formulated at a time when atoms were thought to be like billiards balls, electric current to be some kind of fluid and magnetic fields to be a movement of the aether.

It is possible to reformulate the Laws of electricity and magnetism so that they can be applied to the behaviour of individual electrons. From there, it is possible to derive the laws Faraday and Biot-Savart for the situations in which they give the correct answer.

Most text books on Electricity and Magnetism give a formula for the magnetic field which should surround a moving charge.

$$\vec{B} = \frac{\mu_0\, q\, \vec{v} \wedge \hat{r}}{4\, \pi\, r^2}$$

This magnetic field forms rings about the line of motion of the charge. We might ask what happens if we accelerate the charge in the direction of its velocity. We find that the magnetic field increases in size. If we hold to the idea that magnetic flux is continuous and quantifiable in terms of lines of force, then we should be able to find the magnetic flux emerging from or being generated at the surface of the charge. The problem with this way of looking at a magnetic field is that we normally measure the quantity of magnetic flux as a surface integral.

$$\Phi = \int \vec{B} \cdot d\vec{A}$$

where $d\vec{A}$ is the infinitesimal element of area of a plane which intersects the magnetic field. When we try this for an individual moving charge, we need to take a half plane extending from the line of motion of the charge to infinity and perform the integration in polar coordinates. This gives

$$\begin{aligned} \Phi &= \int_{r_0}^{\infty} \int_0^{\pi} \vec{B} \cdot \hat{r} \;\; r\, d\theta\, dr \\ &= \int_{r_0}^{\infty} \int_0^{\pi} \left(\frac{\mu_0\, q\, \vec{v} \wedge \hat{r}}{4\, \pi\, r^2} \right) \cdot \hat{r} \;\; r\, d\theta\, dr \\ &= \infty \end{aligned}$$

We get a similar result when we try to find the magnetic flux per metre length surrounding a long thin wire, so the effect is not confined to the case of a single charge.

The problem comes when we try to understand this movement of magnetic flux in terms of Faraday's law of induction. When magnetic flux cuts a conductor, it sets up an electric field and we can integrate this around a circuit to get an emf. In our understanding of the propagation of electromagnetic waves, we find that a moving magnetic field generates an electric field in the absence of any conductor. The magnetic flux moves outwards from the surface of the charge and we might expect this movement to generate an electric field. The problem is that any increase in the velocity of the charge will require an infinite amount of magnetic flux to move outwards from the charge generating an infinite electric field. We would expect from the other laws of electricity and magnetism that such an electric field would oppose the acceleration of the charge and because of the infinite strength of the electric field, it should not be possible to accelerate the charge.

The obvious solution to this paradox is to conclude that $\int \vec{B} \cdot d\vec{A}$ is not a very good way of measuring the magnetic field. We need to overcome the problem of the infinite integration and the obvious way to do this is to measure the total energy content of the magnetic field. The energy density of a magnetic field is

$$\begin{aligned} Q_m &= \frac{1}{2\,\mu_0} B^2 \\ &= \frac{1}{2\,\mu_0} \left(\frac{\mu_0\, q\, \vec{v} \wedge \hat{r}}{4\,\pi\, r^2}\right)^2 \\ &= \frac{\mu_0\, q^2\, v^2\, sin^2\,\theta}{32\,\pi^2\, r^4} \end{aligned}$$

where θ is the angle between the line of motion of the charge and the radius vector from the charge. The r^4 term will now ensure a finite integral for the total energy contest of the magnetic field.

$$\begin{aligned} \mathcal{E}_m &= \int Q_m\, d\tau \\ &= \int_{r_0}^{\infty} \int_0^{2\pi} \int_0^{\pi} \frac{\mu_0\, q^2\; v^2\, sin^2\,\theta}{32\,\pi^2\, r^4}\; r^2\, sin\,\theta\, d\theta\, d\phi\, dr \\ &= \frac{\mu_0\, q^2\; v^2}{12\,\pi\, r_0} \end{aligned}$$

Where r_0 is the radius of the charge. We now have a finite measure of the magnetic field surrounding the moving charge in terms of its total energy content and note with interest that, like the kinetic energy of a moving body, it is proportional to v^2.

Our next step is to examine Faraday's law and see if it might be formulated in terms of the energy density of the magnetic field. If a straight conductor of length l is perpendicular to a uniform magnetic and moves with a velocity v perpendicular to the magnetic field and to its length, an emf will be generated in the conductor given by the equation

$$emf = -B\, v\, l$$

The emf is actually due to an electric field E acting over a length l, so we can write

$$E = B\, v$$

If we transform the equation for the energy density to give B

$$Q_m = \frac{1}{2\,\mu_0} B^2 \;\Rightarrow\; B = \sqrt{2\,\mu_0\, Q_m}$$

and we can substitute this to give.

$$E = \sqrt{2\,\mu_0\, Q_m}\;\; v$$

This is an equation which gives the magnitude of the electric field generated by the movement of a conductor through a magnetic field of energy density Q_m in a rather nice situation where the conductor, the magnetic field and the velocity of the wire are mutually perpendicular. When this is not the case, we can use the vector equation

$$\vec{E} = -\vec{B} \wedge \vec{v}$$

$$= \vec{v} \wedge \vec{B}$$

We are more used to expressing Faraday's law in terms of the emf generated around a circuit and be more familiar with the equation

$$dV = \left(\vec{v} \cdot \vec{B}\right) \cdot d\vec{l}$$

but a moments thought will show that the two are equivalent.

The vector form of the equation $E = \sqrt{2\,\mu_0\,Q_m}\;\;v$ may now be written by simply taking into account the directional properties of the equation $\vec{E} = -\vec{B} \wedge \vec{v}$ giving

$$\vec{E} = \sqrt{2\,\mu_0\,Q_m}\;\;v\;\left(\hat{v} \wedge \hat{B}\right)$$

$$= \sqrt{2\,\mu_0\,Q_m}\;\;\vec{v} \wedge \hat{Q}_m$$

since by definition $\vec{Q}_m$ is in the direction of $\vec{B}$.

We have a new way of thinking of a magnetic field. To make it clear that this is so, we will use the term "magnetic energy density flux".

Returning to the acceleration of a charge in the direction of its velocity, we could simply apply the law of conservation of energy to the equation $\mathcal{E}_m = \frac{\mu_0\,q^2\;v^2}{12\,\pi\,r_0}$ and assume that the force F required to accelerate the charge multiplied by the velocity is equal to the rate of increase in energy.

$$F\,v = \frac{d}{dt}\;\mathcal{E}_m$$

$$= \frac{d}{dt}\;\left(\frac{\mu_0\,q^2\;v^2}{12\,\pi\,r_0}\right)$$

$$= \frac{\mu_0\,q^2}{12\,\pi\,r_0}\;2\,v\,\frac{dv}{dt}$$

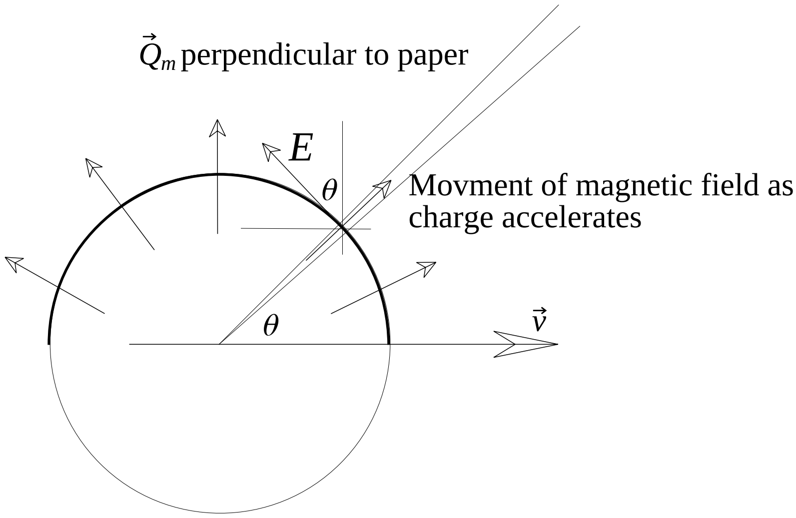
$$F = \frac{\mu_0\,q^2}{6\,\pi\,r_0}\;\;\frac{dv}{dt}$$

$$= \frac{\mu_0\,q^2}{6\,\pi\,r_0}\;\;a$$

We are of cause reminded of Newton's law $F = ma$.

But let us see if we can obtain the same result using the idea that it is the energy content of a magnetic field which is important in calculating induced electric fields. We need to add one more ingredient to our understanding of the nature of a magnetic field. That is the proposition that magnetic flux as defined by its energy content can only move parallel to the electric field of the charge whose motion is generating it. The only reason for making this proposition is that it seems a nice idea which greatly simplifies the analysis and happens to give the right answer. This means that we can now divide the magnetic field into conic elements extending outwards from the surface of the charge parallel to its electric field. In his original description of electric field, Faraday used the idea of imaginary tubes which bounded the electric flux and whose walls were everywhere parallel to it. Conceptually, this is far more useful than the idea of lines of force. If we think of the conic elements which we will use in our integration as the Faraday tubes which describe the electric field of the charge, then the energy content of the magnetic field is constrained to move within them.

The diagram shows a cross section through a charge and a conic element of rectangular cross section $r\,\delta\theta$ by $r \sin\theta\,\delta\phi$ and of area $r^2 \sin\theta\;\delta\theta\,\delta\phi$ defined by spherical polar coordinates symmetric to the line of motion of the charge.



First, we must find the total energy content of the conic element.

$$\delta\mathscr{E}_m = \int_{r_0}^{\infty} \vec{Q}_m\, r^2 \sin\theta\;\delta\theta\,\delta\phi\,dr$$

$$= \int_{r_0}^{\infty} \frac{\mu_0\, q^2\, v^2 \sin^2\theta}{32\,\pi^2\, r^4}\, r^2 \sin\theta\;\delta\theta\,\delta\phi\,dr$$

$$= \int_{r_0}^{\infty} \frac{\mu_0\, q^2\, v^2 \sin^3\theta}{32\,\pi^2\, r^2}\;\delta\theta\,\delta\phi\,dr$$

$$= \frac{\mu_0\, q^2\, v^2 \sin^3\theta}{64\,\pi^2\, r_0}\,\delta\theta\,\delta\phi$$

The rate of increase of $\delta\mathscr{E}_m$ with respect to time is

$$\frac{d}{dt}\,\delta\mathscr{E}_m = \frac{\mu_0\, q^2\;\sin^3\theta}{64\,\pi^2\, r_0}\;\delta\theta\,\delta\phi\;\,2\,v\,\frac{dv}{dt}$$

This is the rate at which energy if flowing into the magnetic field as the charge accelerates in the direction of its velocity. If we divide this by the product of the energy density and the cross sectional area of the conic element at the surface of the charge, then we will get the velocity with which the magnetic energy density flux is emerging from the surface of the charge.

$$v_{flux} = \frac{\dfrac{\mu_0\, q^2\;\sin^3\theta}{64\,\pi^2\, r_0}\;\delta\theta\,\delta\phi\;\,2\,v\,\dfrac{dv}{dt}}{\dfrac{\mu_0\, q^2\, v^2 \sin^2\theta}{32\,\pi^2\, r_0^4}\, r_0^2 \sin\theta\,\delta\theta\,\delta\phi}$$

$$= \frac{r_0}{v}\;\frac{dv}{dt}$$

The geometry of the situation is that the magnetic flux forms concentric rings around the line of motion of the charge. The magnetic energy density flux is moving out of the surface at a velocity of v_{flux} relative to the charge and the electric field will be directed along lines of longitude to an axis in the line of motion of the charge. These are mutually perpendicular so we can use the scalar form of the equation.

$$E = \sqrt{2\,\mu_0\, Q_m}\;\;v_{flux}$$

$$= \sqrt{2\,\mu_0\,Q_m}\ \ \frac{r_0}{v}\ \ \frac{dv}{dt}$$

$$= \sqrt{2\,\mu_0\ \frac{\mu_0\,q^2\,v^2\,sin^2\,\theta}{32\,\pi^2\,r_0^4}}\ \ \frac{r_0}{v}\ \ \frac{dv}{dt}$$

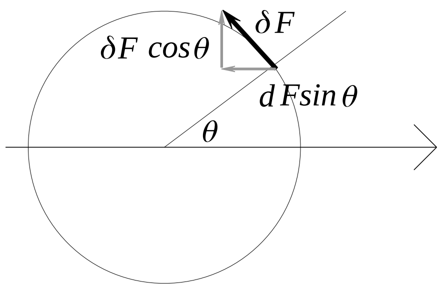
$$= \frac{\mu_0\,q\,sin\,\theta}{4\,\pi\,r_0}\ \ \frac{dv}{dt}$$

This is an electric field acting on an element of the surface of the charge along a line of longitude to an axis in the line of motion. However, we must take into account the fact that the surface of a charge has a finite thickness and that the magnetic energy density flux is generated throughout this surface. Since v_{flux} was calculated by dividing by the energy density of the flux, it remains constant throughout the depth of the surface. The value of the magnetic induction varies linearly from zero at the inner face of the surface to a maximum at the outer face. Thus the electric field is subject to a similar variation and we must introduce a constant of $\frac{1}{2}$ to account for this.

We are now in a position to find the force exerted on the charge by integrating over the surface of the charge.

$$\delta\vec{F}\ =\ \frac{1}{2}\,\vec{E}\,\delta q$$

$\delta\vec{F}$ may be resolved into two components parallel to and perpendicular to the line of motion. In the integration, the latter will sum to zero because of the symmetry of the situation.



We can see that the component of $\delta\vec{F}$ in the direction of motion is $-\,\delta F\,sin\,\theta$.

$$F\ =\ \int -\,\delta F\,sin\,\theta$$

$$= -\int_0^{\pi}\int_0^{2\pi}\ \frac{1}{2}\ \frac{\mu_0\,q\,sin\,\theta}{4\,\pi\,r_0}\ \frac{dv}{dt}\ sin\,\theta\ \ \frac{r_0^2\,sin\,\theta}{4\,\pi\,r_0^2}\ \ d\phi\,d\theta$$

$$= -\frac{\mu_0\,q}{32\,\pi^2\,r_0}\ \frac{dv}{dt}\int_0^{\pi}\int_0^{2\pi} sin^3\,\theta\ \ d\phi\,d\theta$$

$$= -\frac{\mu_0\,q^2}{6\,\pi\,r_0}\ \ \frac{dv}{dt}$$

Which is the desired result. We can conclude that modifying our understanding of the behaviour of magnetic flux and our understanding of the way in which induction works, we are able to get a result consistent with the principle of conservation of energy. Note that we have calculated the force with which the charge resists acceleration whereas before we calculated the force required to accelerate the charge. Hence this result has a minus sign while the other does not.

So far we have confined our attention to the case where the acceleration is in the direction of the line of motion. It should be possible to generalise this to cope with acceleration at any angle to the line of motion. This proved to be a very hard nut to crack.

In principle, acceleration at an angle to the line of motion can be dealt with by resolving it into two components parallel and perpendicular to the line of motion so that it can be dealt with as the sum of a linear acceleration and a centripetal acceleration. We know from classical mechanics that linear acceleration changes the kinetic energy of a body while centripetal acceleration changes the direction of motion and leaves the kinetic energy unchanged. There is a direct equivalent in the case of the acceleration of a charge in that linear acceleration cause a change in the energy content of the magnetic field surrounding the charge, while centripetal acceleration causes a rotation of the magnetic field so that it is always symmetric about the line of motion. It would seem reasonable to suppose that the rotation of the magnetic field is able to generate a force equivalent to centrifugal force, but which ever way one attempts to do the sums, it just does not give the right answer.

One of the problems is that so far as the electric field of a charge is concerned, rotation is meaningless. If a charge were to try to rotate taking its electric field with it, then the result would be the generation of a magnetic field of infinite energy content and finite divergence. If we imagine a single charge at rest, but rotating with an angular velocity $\vec{\omega}$, then $\vec{v} = \vec{\omega} \wedge \vec{r}$ and the magnetic field surrounding the charge should be

$$\vec{B} = \gamma\, \varepsilon_0\, \mu_0 \;\; (\vec{\omega} \wedge \vec{r}) \wedge \vec{E}$$

and we can substitute for $\vec{E}$ giving

$$\vec{B} = \gamma\, \varepsilon_0\, \mu_0 \;\; (\vec{\omega} \wedge \vec{r}) \wedge \frac{q}{4\,\pi\,\varepsilon_0\, r^2}\,\hat{r}$$

$$\vec{B} = \gamma\, \frac{\mu_0\, q}{4\,\pi\, r^2}\;\; (\vec{\omega} \wedge \vec{r}) \wedge \hat{r}$$

The vector $\vec{r}$ may be separated into the product of its magnitude and direction $r\,\hat{r}$ so that the r cancels giving

$$\vec{B} = \frac{\gamma\, \mu_0\, q}{4\,\pi\, r}\;\; (\vec{\omega} \wedge \hat{r}) \wedge \hat{r}$$

The energy density of this magnetic field obeys the inverse square law, and the volume of a spherical shell of thickness δr is proportional to r^2, so the integral of the energy density over volume is infinite. We can also express $\vec{B}$ in Cartesian coordinates and then find its divergence showing it to be non zero. Either rotation is meaningless for a spherical charge, or these two factors combine to firmly fix the orientation of its electric field.

If the electric field is fixed in space with regards to rotation, then the movement of the magnetic energy density flux is also constrained, but adding this factor still does not give the correct result. It is only when we take the directional properties of the magnetic energy density flux into account that we get the right answer.

A charge q has a velocity $\vec{v}$ and an electric field gives it an acceleration $\vec{a}$. If we assume that $\vec{v}$ is not parallel to $\vec{a}$, then we can define Cartesian coordinates such that the X-axis is in the direction of $\vec{v}$. The point $\vec{r}$ has polar coordinates $(r,\ \theta,\ \phi)$ such that its Cartesian coordinates are:

$$x = r \cos\theta \;:\; y = r \sin\theta \cos\phi \;:\; z = r \sin\theta \sin\phi$$

The charge will be surrounded by a magnetic field, which we call its field of motion, given by

$$\vec{B} = \frac{\gamma\, \mu_0\, q}{4\,\pi\, r^2}\; \vec{v} \wedge \hat{r}$$

The acceleration produces a change in the magnetic field given by

$$\frac{d}{dt}\vec{B} = \frac{\gamma\,\mu_0\,q}{4\,\pi\,r^2}\;\frac{d}{dt}\left(\vec{v}\,\wedge\,\hat{r}\right) + \frac{\gamma\,\mu_0\,q}{4\,\pi\,r^2}\;\vec{v}\,\wedge\,\hat{r}\;\frac{d\gamma}{dt}$$

The vector cross product is distributive across vector addition and subtraction while the process by which we prove that $\frac{d}{dt}\vec{v} = \vec{a}$ relies on vector addition and subtraction. We can therefore differentiate the cross product, of $\vec{v}$ by the constant vector $\hat{r}$, with respect to time giving

$$\frac{d}{dt}\vec{B} = \frac{\gamma\,\mu_0\,q}{4\,\pi\,r^2}\;\vec{a}\,\wedge\,\hat{r}$$

This rate of change of the magnetic intensity changes the energy density of the magnetic field and this results in a flow of magnetic energy density flux both into and/or out of of the surface of the charge depending on the relative direction of $\vec{a}$ compared with $\vec{v}$ and on the position on the surface of the charge. The energy density at $\vec{r}$ in its scalar and vector forms is

$$Q_m = \frac{1}{2\,\mu_0}\vec{B}^2 \;:\; \vec{Q}_m = \frac{1}{2\,\mu_0}B\,\vec{B}$$

and we can expand these to get three more equations.

$$Q_m = \frac{1}{2\,\mu_0}\left(\frac{\mu_0\,q}{4\,\pi\,r^2}\right)^2\gamma^2\left(\vec{v}\,\wedge\,\hat{r}\right)^2 \;:\; \vec{Q}_m = \frac{1}{2\,\mu_0}\left(\frac{\mu_0\,q}{4\,\pi\,r^2}\right)^2\gamma^2\;\left|\vec{v}\,\wedge\,\hat{r}\right|\left(\vec{v}\,\wedge\,\hat{r}\right)$$

$$Q_m = \frac{\gamma^2\,\mu_0\,q^2\,v^2\,sin^2\,\theta}{32\,\pi^2\,r^4}$$

The effect of acceleration on a body is twofold. The speed of the body may change and its direction of motion may change. There are two special cases: acceleration parallel to the velocity changes only the speed; acceleration perpendicular to the velocity changes only the direction. When we consider a single spherical charge, the effects on the field of motion are a change in its energy content caused by a change in speed and a rotation caused by to a change in direction. The movement of magnetic energy density flux is constrained so that it can only move parallel to the electric field of the charge. Furthermore, the electric field is fixed in direction and cannot rotate. So rotation of the magnetic field is accomplished by the movement of energy density flux both into and out of the charge.

A simplistic approach might be to differentiate the first equation for Q_m with respect to time to give

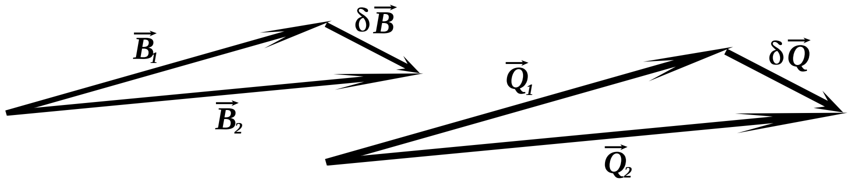
$$\begin{aligned}\frac{d}{dt}Q_m &= \frac{\gamma^2}{2\,\mu_0}\,\frac{d}{dt}\left(\vec{B}^2\right)\\ &= \frac{\gamma^2}{2\,\mu_0}\,2\,\vec{B}\;\cdot\;\frac{d}{dt}\vec{B}\\ &= \frac{\gamma^2}{\mu_0}\left(\frac{\mu_0\,q}{4\,\pi\,r^2}\;\vec{v}\,\wedge\,\hat{r}\right)\cdot\left(\frac{\mu_0\,q}{4\,\pi\,r^2}\;\vec{a}\,\wedge\,\hat{r}\right)\\ &= \frac{\gamma^2\,\mu_0\,q^2}{16\,\pi^2\,r^4}\;\left(\vec{v}\,\wedge\,\hat{r}\right)\cdot\left(\vec{a}\,\wedge\,\hat{r}\right)\end{aligned}$$

This does not however take into account the fact that magnetic energy density flux has vector like properties. If in a small time δt the magnetic intensity changes from $\vec{B}$ to $\vec{B} + \delta\vec{B}$, the energy density vector changes from $\vec{Q}$ to $\vec{Q} + \delta\vec{Q}$. However, the vector addition of $\vec{Q} + \delta\vec{Q}$ must have the same geometry as that of $\vec{B} + \delta\vec{B}$. If we draw vector triangles to represent the addition, they must be similar.

We know that in general

$$\vec{Q}_m = \frac{1}{2\,\mu_0}\,B\,\vec{B}$$

We look to the vector triangle from which the limiting result for $\frac{d}{dt}\,\vec{B}$ is derived and scale it by a factor of $\frac{1}{2\,\mu_0}\,B$ to get the other triangle.



$$\frac{\vec{Q}_{m1}}{\vec{B}_1} = \frac{\delta\vec{Q}_m}{\delta\vec{B}} = \frac{1}{2\,\mu_0}\,B$$

$$\therefore\ \delta\vec{Q} = \frac{1}{2\,\mu_0}\,B\,\delta\vec{B}$$

and this corresponds to the following result.

$$\frac{d}{dt}\vec{Q}_m = \frac{1}{2\,\mu_0}\,B\,\frac{d}{dt}\vec{B}$$

$$= \frac{\gamma^2\,\mu_0\,q^2}{16\,\pi^2\,r^4}\ \left|(\vec{v}\ \wedge\ \hat{r})\right|\ \ (\vec{a}\ \wedge\ \hat{r})$$

This result might appear to violate the principle of energy conservation, but this is not so for a very good reason. Magnetic flux comes in continuous loops and any energy flow into one part of a loop is offset by an equal flow out of another part of it. The change in $\vec{Q}_m$ can be resolved into two components parallel to $\vec{Q}_m$ and perpendicular to it. Energy is conserved when we consider the first of these, but the component perpendicular to $\vec{Q}_m$ changes its direction and not its magnitude. This requires a movement of energy between one side of the field of motion and the other which effects mechanical stress on the charge consistent with the conservation of energy.

At this point, it is best to enter into Cartesian coordinates and simplify the vector product because algebraic expansion would introduce extra sines and cosines of unknown angles. We have

$$\vec{v}\ \wedge\ \hat{r} = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} \cos\theta \\ \sin\theta\cos\phi \\ \sin\theta\sin\phi \end{pmatrix} = \begin{pmatrix} 0 \\ -v\sin\theta\sin\phi \\ v\sin\theta\cos\phi \end{pmatrix}$$

$$\vec{a}\ \wedge\ r = \begin{pmatrix} a_x \\ a_y \\ a_y \end{pmatrix} \wedge \begin{pmatrix} \cos\theta \\ \sin\theta\cos\phi \\ \sin\theta\sin\phi \end{pmatrix} = \begin{pmatrix} a_y\sin\theta\sin\phi - a_z\sin\theta\cos\phi \\ a_z\cos\theta - a_x\sin\theta\sin\phi \\ a_x\sin\theta\cos\phi - a_y\cos\theta \end{pmatrix}$$

$$\left|\vec{v}\ \wedge\ r\right|\ \cdot\ (\vec{a}\ \wedge\ \hat{r}) = \left|\begin{pmatrix} 0 \\ -v\sin\theta\sin\phi \\ v\sin\theta\cos\phi \end{pmatrix}\right| \begin{pmatrix} a_y\sin\theta\sin\phi - a_z\sin\theta\cos\phi \\ a_z\cos\theta - a_x\sin\theta\sin\phi \\ a_x\sin\theta\cos\phi - a_y\cos\theta \end{pmatrix}$$

The magnitude $\left|\vec{v}\ \wedge\ r\right|$ is easy enough to work out as $v\sin\theta$ giving

$$|\vec{v} \wedge r| \cdot (\vec{a} \wedge \hat{r}) = v \sin\theta \begin{pmatrix} a_y \sin\theta \sin\phi - a_z \sin\theta \cos\phi \\ a_z \cos\theta - a_x \sin\theta \sin\phi \\ a_x \sin\theta \cos\phi - a_y \cos\theta \end{pmatrix}$$

Which gives

$$\frac{d}{dt}\vec{Q}_m = \frac{\mu_0 q^2}{16 \pi^2 r^4} v \sin\theta \begin{pmatrix} a_y \sin\theta \sin\phi - a_z \sin\theta \cos\phi \\ a_z \cos\theta - a_x \sin\theta \sin\phi \\ a_x \sin\theta \cos\phi - a_y \cos\theta \end{pmatrix}$$

At this point, we are in danger of running into complications. We really need this vector in the form of a magnitude times a unit vector, but this expression is rather messy. So we will delay dealing with this problem by making a substitution.

$$Let\ \vec{\lambda} = \lambda \hat{\lambda} = v \sin\theta \begin{pmatrix} a_y \sin\theta \sin\phi - a_z \sin\theta \cos\phi \\ a_z \cos\theta - a_x \sin\theta \sin\phi \\ a_x \sin\theta \cos\phi - a_y \cos\theta \end{pmatrix}$$

Then we can write

$$\frac{d}{dt}\vec{Q}_m = \frac{\mu_0 q^2}{16 \pi^2 r^4} \lambda \hat{\lambda}$$

We need the rate of change of the magnetic energy density which is

$$\frac{d}{dt} Q_m = \frac{\mu_0 q^2}{16 \pi^2 r^4} \lambda$$

We need to divide the space surrounding the charge in such a way as to facilitate integration. We do this by dividing it into conic elements extending outwards from the sphere with rectangular cross section $\delta\phi$ by $\sin\theta\ \delta\theta$. We need to integrate this over r to find the rate of flow of energy density flux into the conic element. The area of cross section of the conic element is $\delta A = r^2 \sin\theta\ \delta\theta\ \delta\phi$

$$\frac{d}{dt}\delta\mathscr{E}_m = \int \frac{d}{dt} Q_m\, \delta A\, dr$$

$$= \int_{r_0}^{\infty} \frac{\mu_0 q^2}{16 \pi^2 r^4} \lambda\ r^2 \sin\theta\ \delta\theta\ \delta\phi\ dr$$

$$= \frac{\mu_0 q^2}{16 \pi^2 r_0} \lambda\ \sin\theta\ \delta\theta\ \delta\phi$$

To find the velocity with which magnetic energy density flux is flowing from the surface of the charge, we must divide by the energy density times the area of the surface of the charge within the conic element.

$$v_{flux} = \frac{\dfrac{\mu_0 q^2}{16 \pi^2 r_0} \lambda\ \sin\theta\ \delta\theta\ \delta\phi}{\dfrac{\mu_0 q^2 v^2 \sin^2\theta}{32 \pi^2 r_0^4}\ r_0^2 \sin\theta\ \delta\theta\ \delta\phi}$$

$$= \frac{2 r_0 \lambda}{v^2 \sin^2\theta}$$

Comparing this with the result at this stage in our calculation of the force resisting acceleration in the direction of the velocity, we note the terms 2γ and $v^2 \sin^2\theta$ and hope all is well. We do not know what γ is, but we know it depends on v and θ. We will need this velocity as a vector and since it is a flow out of the surface of the charge

$$\vec{v}_{flux} = \frac{2\, r_0\, \lambda}{v^2 \sin^2\theta}\,\hat{r}$$

Let us recall what we are doing and why we must do it. Magnetic energy density flux has directional properties. We are changing the magnetic energy density flux with the conic element. We need to change both its energy density and its direction. To effect the change in direction, we add (or subtract) magnetic energy density flux with a directional property which is at at an angle to the energy density flux which is already in the conic element. The energy density of the flux which we add is the energy density at the surface of the charge. We therefore combine the magnitude $\sqrt{2\,\mu_0\,Q_m}$ and the direction $\hat{\gamma}$ in our new form of Faraday's law to calculate the the electric field experienced by the surface of the charge.

$$\begin{aligned}
\vec{E} &= \left(\sqrt{2\,\mu_0\,Q_m}\;\;\hat{\lambda}\right) \wedge \vec{v}_{flux} \\
&= \left(\sqrt{2\,\mu_0\;\frac{\mu_0\,q^2\,v^2 \sin^2\theta}{32\,\pi^2\,r_0^4}}\;\;\hat{\lambda}\right) \wedge \left(\frac{2\,r_0\,\lambda}{v^2 \sin^2\theta}\,\hat{r}\right) \\
&= \frac{\mu_0\,q\,v \sin\theta}{2\,\pi\,r_0\;v^2 \sin^2\theta}\;\;\lambda\,\hat{\lambda}\;\;\wedge\;\hat{r}
\end{aligned}$$

Now we find that λ and $\hat{\lambda}$ are reunited and we can substitute for $\lambda\,\hat{\lambda}$.

$$\begin{aligned}
\vec{E} &= \frac{\mu_0\,q\,v \sin\theta}{2\,\pi\,r_0\;v^2 \sin^2\theta}\;v \sin\theta \begin{pmatrix} a_y \sin\theta \sin\phi - a_z \sin\theta \cos\phi \\ a_z \cos\theta - a_x \sin\theta \sin\phi \\ a_x \sin\theta \cos\phi - a_y \cos\theta \end{pmatrix} \wedge\;\hat{r} \\
&= \frac{\mu_0\,q}{2\,\pi\,r_0} \begin{pmatrix} a_y \sin\theta \sin\phi - a_z \sin\theta \cos\phi \\ a_z \cos\theta - a_x \sin\theta \sin\phi \\ a_x \sin\theta \cos\phi - a_y \cos\theta \end{pmatrix} \wedge\;\hat{r}
\end{aligned}$$

The force felt by the charge acting on the surface within the conic element is

$$\delta\vec{F} = \frac{1}{2}\,\vec{E}\,\delta q$$

where the factor $\frac{1}{2}$ is the penetration coefficient. This is needed because the surface of the charge has a finite thickness. Energy density flux is generated throughout the thickness of the surface. We have used the value of $\vec{E}$ at the outer face of the surface, but this gives us a maximum value of $\vec{E}$ and we need to average it over the thickness of the surface.

$$\begin{aligned}
\delta\vec{F} &= \frac{1}{2}\;\frac{\mu_0\,q}{2\,\pi\,r_0} \begin{pmatrix} a_y \sin\theta \sin\phi - a_z \sin\theta \cos\phi \\ a_z \cos\theta - a_x \sin\theta \sin\phi \\ a_x \sin\theta \cos\phi - a_y \cos\theta \end{pmatrix} \wedge\;\hat{r}\;\;\frac{q\,r_0^2 \sin\theta\;\delta\theta\;\delta\phi}{4\,\pi\,r_0^2} \\
&= \frac{\mu_0\,q^2 \sin\theta}{16\,\pi^2\,r_0} \begin{pmatrix} a_y \sin\theta \sin\phi - a_z \sin\theta \cos\phi \\ a_z \cos\theta - a_x \sin\theta \sin\phi \\ a_x \sin\theta \cos\phi - a_y \cos\theta \end{pmatrix} \wedge \begin{pmatrix} \cos\theta \\ \sin\theta \cos\phi \\ \sin\theta \sin\phi \end{pmatrix} \delta\theta\;\delta\phi
\end{aligned}$$

$$= \frac{\mu_0 q^2}{16 \pi^2 r_0} \begin{pmatrix} a_z \sin^2\theta \cos\theta \sin\phi - a_x \sin^3\theta \sin^2\phi - a_x \sin^3\theta \cos^2\phi + a_y \sin^2\theta \cos\theta \cos\phi \\ a_x \sin^2\theta \cos\theta \cos\phi - a_y \sin\theta \cos^2\theta - a_y \sin^3\theta \sin^2\phi + a_z \sin^3\theta \sin\phi \cos\phi \\ a_y \sin^3\theta \sin\phi \cos\phi - a_z \sin^3\theta \cos^2\phi - a_z \sin\theta \cos^2\theta + a_x \sin^2\theta \cos\theta \sin\phi \end{pmatrix} \delta\theta\, \delta\phi$$

$$\vec{F} = \frac{\mu_0 q^2}{16 \pi^2 r_0} \int_0^{2\pi} \int_0^{\pi} \begin{pmatrix} a_z \sin^2\theta \cos\theta \sin\phi - a_x \sin^3\theta \sin^2\phi - a_x \sin^3\theta \cos^2\phi + a_y \sin^2\theta \cos\theta \cos\phi \\ a_x \sin^2\theta \cos\theta \cos\phi - a_y \sin\theta \cos^2\theta - a_y \sin^3\theta \sin^2\phi + a_z \sin^3\theta \sin\phi \cos\phi \\ a_y \sin^3\theta \sin\phi \cos\phi - a_z \sin^3\theta \cos^2\phi - a_z \sin\theta \cos^2\theta + a_x \sin^2\theta \cos\theta \sin\phi \end{pmatrix} d\theta\, d\phi$$

This integration is not as hard as it looks because a lot of the integrals of individual terms are zero. It simplifies to

$$\vec{F} = \frac{\mu_0 q^2}{16 \pi^2 r_0} \int_0^{2\pi} \int_0^{\pi} \begin{pmatrix} -a_x \sin^3\theta \\ -a_y \sin\theta \cos^2\theta - a_y \sin^3\theta \sin^2\phi \\ -a_z \sin^3\theta \cos^2\phi - a_z \sin\theta \cos^2\theta \end{pmatrix} d\theta\, d\phi$$

$$= \frac{\mu_0 q^2}{16 \pi^2 r_0} \begin{pmatrix} -\frac{8\pi}{3} a_x \\ -\frac{8\pi}{3} a_y \\ -\frac{8\pi}{3} a_z \end{pmatrix}$$

$$= -\frac{\mu_0 q^2}{6 \pi r_0} \vec{a}$$

We find that effecting the change in the field of motion generates a force proportional to and opposing the acceleration. Thus a simple spherical charge with no other properties has an inherent electromagnetic reaction to acceleration. The charge has the property which we normally call inertial mass. We may write a familiar equation.

$$\vec{F} = m \vec{a}$$

where $\vec{F}$ is now the force required to accelerate the charge and

$$m = \frac{\mu_0 q^2}{6 \pi r_0}$$

The inertial mass of a pure charge is thus dependent on its charge and its radius.

This result is fully consistent with the idea that the kinetic energy of the charge is stored in its field of motion.

$$\mathcal{E} = \int Q_m \, d\tau$$

$$= \frac{\mu_0 q^2 v^2}{12 \pi r_0}$$

$$= \frac{1}{2} m v^2$$

So we find that we can account for the generation of a force resisting that acceleration, and of the correct magnitude to account for the changes in the energy stored in the magnetic field which surrounds a charge by virtue of its velocity. To do this, we have had to refine our understanding of the way magnetic flux behaves. Instead of measuring the bulk of a magnetic field as the integral over a surface of magnetic induction through that surface, we take the volume integral of the energy density of the field. When a field changes in size, it is the flow of energy into and out of the field which is calculated. We still need to take into account the directional properties of magnetic flux when considering it as an energy density field. We need also to specify that movement of energy density flux can only take place parallel to the electric field of the charge.

When we consider real electrons and take the charge on an electron together with the Bohr radius of the electron and Einstein's $E = mc^2$, we can calculate that the mass equivalent of the energy in the electric field of an electron is

$$m_e = \frac{\mu_0 q^2}{8 \pi r_0}$$

The similarity in the two suggests that we are close to explaining the inertial mass of electrons as a purely electromagnetic phenomena in spite of the fact that many now believe the electron to be point sized.

If m_e is the mass of an electron and e the charge of the the electron, then a spherical charge $-e$ of radius $\frac{\mu_0 q^2}{6 \pi m_e}$ will have the same inertia as an electron.

Similarly a spherical charge $\frac{2}{3} e$ of radius $\frac{2 \mu_0 q^2}{9 \pi m_n}$ will have the same inertia as an up quark and a spherical charge $-\frac{1}{3} e$ of radius $\frac{\mu_0 q^2}{18 \pi m_n}$ will have the same inertia as a down quark if m_n is a little more than the mass of a neutron.

Matter built of such charges will exhibit a reduction in inertia similar to the loss in mass due to binding energy. This is because the formation of the outer regions of the fields of motion of the individual charges will be affected by the influence of the electric fields of other charges.

We know from mechanics that the kinetic energy of a group of bodies is equal to the sum their kinetic energies calculated in the centre of mass reference frame and the kinetic energy of a single body, of mass equal to their total mass, moving at the velocity of the centre of mass. With appropriate expansion and organisation of terms, we can show this principle applies to groups of groups of bodies. We can thus conceive that a body consisting of a lump of matter consisting of atoms built from such charges would be able store the kinetic energy of the body plus the thermal energy of the body plus the internal energy of the atoms in the fields of motion of the individual charges. Such a body would exhibit the property of inertial mass and obey Newtons laws of motion.